On the Propagation of Electromagnetic Waves in an Inhomogeneous Medium

by

Anjum Ali

A Thesis Presented to the

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This thesis is dedicated to my parents

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ABSTRACT

The propagation of electromagnetic waves in inhomogeneous dielectric media (with rectangular as well as cylindrical geometry) having a perturbed-parabolic refractive index has been analysed in detail. Evaluation of the field distributions and the corresponding propagation constants has been carried out for the special case of uniformity in one transverse direction. A general polynomial type of perturbation on the parabolic refractive index has been treated. A series solution in terms of the Hermite functions has been set up for the perturbed fields in the case of an infinite medium in rectangular coordinates, while a power series, in terms of the radial coordinate, has been used for the case of a graded-core optical fiber in cylindrical coordinates. Special examples of linear, cubic and quartic perturbation have been solved numerically using computer techniques and important results have been reported.

INTRODUCTION

The study of wave propagation in inhomogeneous media is of great interest in physics as well as engineering and a considerable amount of work has been reported in the technical literature in the context of

(a) Focusing of waves by naturally occuring ducts in tropospheric communication.

(b) Radio wave propagation in the ionosphere.

(c) Sound wave propagation in sea water and in the atmosphere.

(d) Fiber optics and optical communication (including optical waveguides).

(e) Propagation in plasmas.

(f) Analysis of semiconductor lasers.

(g) Design of microstriplines.

(h) The class of problems which relate to the study of the discrete energy levels of a Harmonic Oscillator in Quantum Mechanics.

Rigorous closed form analytic solutions for the field functions are only possible for some special mathematical forms of permittivity, potential or other characteristic profiles. The general case, for which no closed form analytic solutions are known, is treated by some approximate technique or another. Recently a variety of computer oriented numerical methods have also been used.

In this work two approaches have been used in the treatment of electromagnetic wave propagation in focusing media in which the refractive index profiles depart from the quadratic. The perturbation on the parabola is taken in the form of a polynomial. In the first case, an infinite medium is considered and a perturbation method is used, while in the second case a finite medium with a cylindrical geometry is considered and a numerical method is applied.

Chapter one starts with a review of the problem of propagation in inhomogeneous media. Different aspects of the problem along with important references are mentioned. A summary of the developments in the field of optical communications is also included.

Chapter two develops some basic but important concepts related to wave propagation in general and guided wave propagation in particular. It also contains fundamental equations, which are relevant to the thesis.

Chapter three contains a detailed discussion of the first approach, in which the study is basically centered about the tightly bound modes, in an infinite, inhomogeneous dielectric medium with refractive index variations of the form of a perturbed parabola. The refractive index decreases with distance from the origin in one transverse direction. Cartesian coordinates have been used and the perturbed fields are expressed as infinite series in terms of Hermite functions, which are solutions for the unperturbed case. Particular examples of linear, cubic and quartic perturbations have been treated.

Chapter four discusses a numerical method to investigate the propagation of electromagnetic waves along a radially inhomogeneous optical fiber. The solution within the core is set up in terms of a power series, whereas the solution in the homogeneous cladding is expressed in terms of the Modified Bessel function. Propagation constants and field distributions for propagating TE and TM modes have been evaluated for the linear, cubic and quartic cases, and are included in this chapter.

Finally in chapter five, the conclusions from this study, along with recommendations for future investigations are listed.

1.0 A REVIEW OF THE PROBLEM

1.1 PROPAGATION IN INHOMOGENEOUS MEDIA

An inhomogeneous medium is one in which the dielectric constant 'ϵ' (or the refractive index 'n') is a function of space. The problem of determining the propagation characteristics of electromagnetic waves in such a medium is very important. The reason is that there is a strong resemblance between this problem and other problems in physics and engineering for example acoustic wave propagation, quantum harmonic oscillator etc. Because of the fact that the underlying mathematical treatment is similar in all these cases, a considerable amount of attention has been focused on such problems in the past.

Electromagnetic wave propagation in inhomogeneous media has been analysed extensively by many authors to determine the earth and ionosphere effects on radio waves |1,2|. Among the first people to study propagation in the troposphere were Pekeris |3| and Booker and Walkinshaw |4|. A naturally occuring duct can be formed by the rapid decrease in the moisture content of the atmosphere in the first few meters above the surface of the earth or sea |5|. This variation is due to the high rate of evaporation and diffusion in this region. The concurrent decrease in the refractive index may cause electromagnetic energy, which is radiated within the duct, to be trapped if certain physical requirements are satisfied |6|. In addition to such regions of earth attached ducts, trapping of energy can occur in elevated regions of inhomogeneity |7,8|. Elevated or whispering-gallery ducts have been found to cause long-distance radio communication at VHF and higher frequencies.

In the large majority of investigations the troposphere is considered to be a layered structure i.e., a structure in which the inhomogeneity occurs along one dimension only. The method of representation of the inhomogeneity have also been divided into two types: one that tries to represent the inhomogeneity by a suitable functional profile continuous over the layers, and another that resorts to represent the inhomogeneity by piece-wise linear segments. For xample Chang |9| used a linear segmented numerical method for refractive index profile and calculated the wave modes. Pappert and Goodhart |10| investigated the propagation in off-shore ducting environments by using a trilinear refractive index model for the troposphere. They considered ground-based as well as earth-detached ducts produced by elevated layers at about 600 and 2000 feet respectively by assuming the layers to be stationary and horizontally homogeneous. Joseph and Smith |6| assumed simple exponential profiles and solved the wave equation exactly by the well-established methods of mode theory.

Two dimensional variations in the refractive index have also been treated in the literature. Sodha et. al., |11| have considered a permittivity profile of the form $\varepsilon = \varepsilon_0 - \varepsilon_2(z) \cdot y^2$. In this case the medium is both layered and laterally focusing. They then reviewed a number of $\epsilon_2(z)$ functions (linear, straight wedge-cut, parabolic etc.) yielding analytic solutions but did not impose boundary conditions.

A number of analytical and numerical techniques have been used in the analysis of wave propagation. Approximate analytical methods include the perturbation techniques, the variational techniques and the phase-integral or WKB method. The propagation characteristics of a waveguide mode in the troposphere near the ground were studied by simple phase-integral methods by Hayes |12| over a range of frequencies in which the mode changes from tightly bound to leaky. Based on the original work of Booker and Walkinshaw |4|, he applied the method to a troposphere with a parabolic distribution for the square of the modified refractive index. A generalized WKB approach, transforming Maxwell's Equations for inhomogeneous isotropic media into coupled ordinary differential equations, was used by Bahar |13| to derive expressions for the reflection and transmission coefficients and the characteristic surface impedance as functions of the transverse wave number. Numerical solutions can be obtained by direct integration of the differential equations. Initial value techniques, like the Runge-Kutta method, or boundary value techniques may also be applied |14|.

The theory of inhomogeneous media also applies to the study of microstriplines. A microstrip is the most commonly used transmission structure in microwave integrated circuits. It consists of a narrow conducting strip on one side of a dielectric substrate and a completely metallized surface on the other |15|. The metallized surface serves as a ground plane. Microwave energy propagates in both the dielectric substrate below the strip and in the air region in its vicinity. The problem of determining the electromagnetic fields about conductors surrounded by various dielectric materials, such as in the analysis of transmission line configurations, computer circuit boards, electro-optic effects in crystals etc. was described by Gear |16|. Other work related to microstrips is |17,18|.

Examples of sound wave propagation in sea water are concerned with the influence of the variation of the speed of sound with depth on the acoustic field |19,20|. This in effect is equivalent to the variation of permittivity or refractive index in case of electromagnetic wave propagation in inhomogeneous media.

Other studies of wave propagation in inhomogeneous media are found in propagation through plasmas |21| and semiconductor lasers |22-27|.

1.2 OPTICAL COMMUNICATIONS

During the past several years, a rapid development has taken place both in technological know-how and theoretical understanding of optical-waveguides and along with it, a new variety of fiber structures has emerged. Thus optical communications has evolved from a mere research activity to a point where systems implementation has touched the limits of certainity. This success was assured when, in 1969, the first 20 dB/km optical fiber (known as the SELFOC— or self-focusing fiber) was demonstrated |28|. Before that, typical fiber attenuations ranged in the thousands of dB/km range, a loss level which precluded consideration of optical waveguides for data transfer. With optical fiber losses now below 1 dB/km |29|, and with the development of suitable solid state diode light sources and detectors, there are no remaining technological barriers which will prevent fiber optical transmission systems from finding widespread applications in the near future. Technological advantages offered by fiber implementation of data transfer systems guarantee the use of optical fibers at least in specialized applications, regardless of economic issues.

This optimistic future for optical communications was achieved only after many different and various approaches to utilize light as an information carrier had been attempted. With the invention of the laser in early 1960's, exploitation of the immense information-carrying capacities promised by optical frequency radiation was widely envisioned. However, progress was limited by two factors: components and the transmission media. Component progress during the sixties was continuous with several types of optical transmitters (lasers and LED's), modulators and photo-detectors being developed. Suitable sources with adequate power existed at the end of the sixties for transmission in low-loss media. However a transmission medium with acceptable transmission characteristics did not exist. Transmission in the open atmosphere was long recognized to be unacceptable and unreliable for light transmission because adverse weather conditions significantly degraded system performance. A controlled atmosphere using evacuated pipes was postulated as a means to circumvent this problem, but in order to maintain beam quality, periodic refocusing was required. The most promising approach employed a conduit filled with a gas which had a radial temperature gradient to refocus the optical beam. The radial temperature profile was obtained using suitably placed interactive servo-controlled heater elements which gave rise to a radial gas-density gradient (and thus a radial refractive index gradient). Radially graded refractive indices provided a continuous beam refocusing and in a sense formed a waveguide. Technical feasibility was established for this approach; however, practical considerations of index profile control and size precluded system utilization of this approach.

The realization of the SELFOC fiber changed the outlook for optical communications by providing for a stable, flexible and a low-loss transmission medium. It also offered some other properties which are as follows |28|:

(1) Laser beam transmission without band-limitation and waveform distortion.

(2) Simultaneous transmission of a number of laser beams without mutual interference.

(3) Conservation of polarization plane for laser beam transmission.

(4) Optical image transmission previously impossible.

(5) Realization of lenses with tiny aperture and ultra-short focal length.

Cylindrical fibers of step or graded-index were quickly perfected, with transmission losses now being reduced to below 1 dB/km and tensile strengths exceeding 10^5 psi/km |29|. As a result of this, long distances in the range of 40-60 km between repeaters are becoming feasible. At these lengths, fiber dispersion will limit the available data capacity to below 20 MHz per channel. However, the enormous bandwidths of single mode fibers (~ 100 GHz) can be used to eliminate these data capacity limitations.

Current developmental emphasis in the fiber optical communications field is placed on environmentally qualifying components such as fiber cables, sources, detectors etc., and on prototype system demonstrations. These demonstrations are designed to test technical feasibility and to develop techniques and experience. Economic viability is largely limited to projections at the present time, however it is widely hoped that fibers will compete with metallic cables in certain frequency ranges, when large scale fiber production is achieved.

One of the transmission line configurations employed in optical communication systems is the optical waveguide consisting of a core and a cladding, the cladding being a mantle consisting of a material whose refractive index is lower than that of the core. Depending on the speed of pulling the optical fiber, the boundary between the core and mantle can be either abrupt or a transitional region may occur. In the latter case, we are dealing with a radially inhomogeneous optical waveguide. For single-mode as well as for multi-mode surface wave operation, the electromagnetic field vectors will decay to a negligibly small value at the outer radius of the guide. In the calculations, one may therefore assume the cladding to extend to infinity.

1.3 PRESENT STATUS OF THE PROBLEM

Because of the difficulty in carrying out a rigorous analysis of the natural modes of an inhomogeneous medium, it is perhaps not surprising that a wide variety of approximate treatments has been reported. This difficulty arises because the problem is described by a fourth-order differential equation (or equivalently a pair of coupled second-order equations) with variable coefficients. In electromagnetics, there are relatively few systems of physical interest for which the wave equations can be solved exactly.

Approximate methods are therefore expected to play an important part in virtually all applications of the theory. This enhances, rather than diminishes, the importance of those problems for which exact solutions can be found since exact solutions are often used as starting points for approximate calculations. They may also help to establish limits of validity for various approximation methods.

Thus two classes of approximations have been entertained. In the first class, the pair of coupled second-order equations for the longitudinal components of the electric and magnetic field vectors are reduced, by properly neglecting terms, to some uncoupled second-order problem, whose solutions have been studied extensively and can preferably be written down explicitly. The most popular of these is the scalar wave approximation |30| which neglects the gradient of the logarithm of the dielectric constant from the Maxwell's Equations. It has been shown by |11| that the contribution of the $\nabla\varepsilon$ term in these equations will be negligible if $\frac{1}{k^2} \nabla^2 (\ell n\ \varepsilon) << 1$. (where $k^2 = \omega^2\mu\varepsilon$). The approximation involved here implies that the index of refraction varies only very slightly over the distance of one wavelength. This condition is true for most situations of interest and particularly for a square-law medium |31|. The electric and magnetic field vectors then separately satisfy the reduced wave equation. This equation has been solved exactly for an unbounded square-law medium |32|, and approximately for a number of refractive index profiles by a variational method |33| and by asymptotic methods |34|. Because asymptotic methods are awkward to apply in practice, particularly when a turning point problem arises and when boundary conditions are to be inserted, these methods have never been applied beyond the zero-th order or WKB approximation, and almost always to infinite media. Conventional application of asymptotic methods is, in effect, restricted to situations in which the refractive index varies slowly within the distance of a wavelength, and in which the discontinuities in index are unimportant.

In another type of approximation (related to the first class), the effective refractive indices of the modes under study are assumed to lie close to the refractive index at the guide center, and that the dielectric function has a slow variation locally. Hence although the coupling between the longitudinal fields is essentially maintained, the system is reduced to a second-order problem. These conditions evidently restrict the applicability of this theory to modes whose fields are tightly bound. Using this approach, Kurtz and Streifer |35| developed analytic solutions for an unbounded medium with quadratic dielectric profile. Later, |36| extended this calculation to treat a cladded fiber (also with a quadratic profile).

In the second class of approximations, Maxwell's equations are kept intact, but different numerical methods are used to solve them. These include the step-by-step initial value techniques employed by |37|. The approach used by |38| reduced the problem to that of a sequence of homogeneous cylindrical shells by replacing the index distribution by a stair case function. The solutions for the fields in each shell satisfy the boundary conditions at the interface between two adjacent shells. Another method employs Green's function techniques to rigorously formulate the electromagnetic fields in terms of integral equations |39| which are solved using standard iterative techniques. In 1973, Dil and Blok |40| numerically calculated the dispersion curves in the cut-off region for parabolic and diffusion-type dielectric profiles. In the next year, Vassel |41| studied the characteristics of axially symmetric propagating modes of an optical fiber (with either a parabolic or a hypersecant profile) using a combination of different numerical techniques. He employed the step-by-step integration in cylindrical shells where the solution is of oscillatory type, and a two-point-boundary-value technique in shells where the solution is of exponential type. The process of matching the core solution, at the core boundary, to the cladding solution resulted in the evaluation of the propagation constants of different propagating modes.

2.0 FUNDAMENTALS OF WAVE PROPAGATION

2.1 WAVE PROPAGATION

Electromagnetic waves may be transmitted along some form of guided structures and are called guided waves. Alternatively, they may be propagated through free space and are therefore unguided waves. In both cases, energy is carried by the electric and magnetic fields associated with the wave. In this chapter, some basic concepts and relevant equations related to wave propagation will be developed.

2.2 MAXWELL'S EQUATIONS

The solution of any electromagnetic problem requires the satisfaction of Maxwell's Equations which are given below in MKS units |44,45|.

$$\nabla \times \overline{E} = -\frac{\partial}{\partial t} \overline{B} \qquad (2\text{-}2\text{-}1)$$

$$\nabla \times \overline{H} = \frac{\partial}{\partial t} \overline{D} + \overline{J} \qquad (2\text{-}2\text{-}2)$$

$$\nabla \cdot \overline{D} = \rho \qquad (2\text{-}2\text{-}3)$$

$$\nabla \cdot \overline{B} = 0 \qquad (2\text{-}2\text{-}4)$$

where

$\overline{E}$: Electric field strength

$\overline{H}$: Magnetic field strength

$\overline{D}$: Electric flux density

$\overline{B}$: Magnetic flux density

$\overline{J}$: Current density

and ρ represents the total charge per unit volume (which can be a sum of the bound and free charges).

In addition there are three relations that concern the characteristics of the medium in which the fields exist

$$\overline{D} = \varepsilon \overline{E} \tag{2-2-5}$$

$$\overline{B} = \mu \overline{H} \tag{2-2-6}$$

$$\overline{J} = \sigma \overline{E} \tag{2-2-7}$$

where ε, μ and σ are the permittivity, permeability and conductivity of the medium respectively.

The propagation of waves in inhomogeneous media involves a very wide range of possibilities which arise mainly when ε is a function of position. In this work the media considered have $\varepsilon = \varepsilon\ (x)$ and $\varepsilon = \varepsilon\ (r)$, i.e., having plane or cylindrical layers. In both cases μ is a constant. In addition, in all the subsequent work, the field quantities will be assumed to vary with time according to the complex exponential function $e^{j\omega t}$, where ω is the radian frequency. This restriction is justified because extremely monochromatic radiators are often used in practice and as a result, the radiated waves are close to harmonic. Moreover there is little loss of generality in using such a time function since any physically realizable periodic time variation can be decomposed into a spectrum of such functions by means of the Fourier Series. Thus any component of the electric or magnetic fields can be expressed by equations of the following general form

$$f(x,y,z,t) = g(x,y,z)\ e^{j\omega t} \tag{2-2-8}$$

With the above assumed time variation, all time derivatives may be replaced by $j\omega$, and so the first two Maxwell's Equations take up the following form:

$$\nabla \times \overline{E} = -j\omega\overline{B} = -j\omega\mu\overline{H} \tag{2-2-9}$$

$$\nabla \times \overline{H} = j\omega\overline{D} + \sigma\overline{E} = (\sigma + j\omega\varepsilon)\ \overline{E} \tag{2-2-10}$$

2.3 THE WAVE EQUATIONS FOR A PERFECT DIELECTRIC

In this section, Maxwell's Equations will be used to derive differential equations relating the electric and magnetic field strengths, $\overline{E}$ and $\overline{H}$, for a non-conducting, homogeneous, isotropic and source-free dielectric medium (eg., vacuum). These differential equations can then be solved simultaneously to determine the laws which both $\overline{E}$ and $\overline{H}$ must obey. These laws are known as the Wave Equations.

Taking the curl of Eqn. (2-2-1) and differentiating (2-2-2) w.r.t. time (remembering that $\sigma = 0$), gives

$$\nabla \times \nabla \times \overline{E} = -\mu \, \nabla \times \frac{\partial \overline{H}}{\partial t} \tag{2-3-1}$$

$$\nabla \times \frac{\partial \overline{H}}{\partial t} = \varepsilon \frac{\partial^2}{\partial t^2} \overline{E} \tag{2-3-2}$$

Combining these two equations

$$\nabla \times \nabla \times \overline{E} = -\mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \tag{2-3-3}$$

Using the vector identity (A-1-1), and the fact that $\nabla \cdot \overline{E} = \frac{1}{\varepsilon} \nabla \cdot \overline{D} = 0$, (since $\rho = 0$), yields

$$\nabla^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \tag{2-3-4}$$

This is the wave equation for $\overline{E}$ —— a law that $\overline{E}$ must obey. Similar steps show that $\overline{H}$ also obeys the same law i.e.,

$$\nabla^2 \overline{H} = \mu\varepsilon \frac{\partial^2 \overline{H}}{\partial t^2} \tag{2-3-5}$$

2.4 UNIFORM PLANE WAVES

Consider a dielectric medium in cartesian coordinates. The wave Eqns. (2-3-4) and (2-3-5) hold for each component of the electric and magnetic field vectors —— that is, each one of E_x, E_y, E_z, H_x, H_y and H_z satisfy the following scalar wave equation

$$\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\Psi \qquad (2\text{-}4\text{-}1)$$

with $$v^2 = 1/\mu\varepsilon \qquad (2\text{-}4\text{-}2)$$

having the physical significance of the velocity of light in the medium with relative dielectric constant $\varepsilon_r = \frac{\varepsilon}{\varepsilon_o}$ and refractive index $n = \sqrt{\varepsilon_r}$.

The significance of the wave equations can easily be appreciated if it is realized that every function of the general form

$$\Psi = g\,(t - \frac{1}{v}\,\bar{n}\cdot\bar{r}) \qquad (2\text{-}4\text{-}3)$$

is a solution of these equations provided the second derivative of 'g' exists. The components of the vector $\bar{r}$ are the coordinates of the point at which the field is being observed; $\bar{n}$ is a unit vector.

Equation (2-4-3) represents plane waves in a space which is homogeneously filled with a medium of relative permittivity ε_r. To demonstrate this, it is necessary to consider a certain fixed value of its argument

$$u = t - \frac{1}{v} (\bar{n} \cdot \bar{r}) \tag{2-4-4}$$

For any given value of u, the function has a corresponding value g(u). The value 'u = constt.' is realized for a fixed value of time t on a plane defined by the relation $\bar{n} \cdot \bar{r}$ = constant. The vector $\bar{n}$ is directed perpendicular to the plane. The same value of the function is therefore to be found over an infinite plane in space. To watch how this particular value of the function behaves as time advances, we allow t to change by Δt and the vector $\bar{r}$ by $\Delta\bar{r}$ in such a way that u remains constant. The relation between the increment of time and the change of the position vector which keeps u unchanged is given by

$$\bar{n} \cdot \Delta\bar{r} = v\,\Delta t \tag{2-4-5}$$

The endpoint of $\Delta\bar{r}$ lies again on a plane. The vector $\bar{n}$ is the normal to both the original and the displaced plane. The plane described by (2-4-4) has moved in the direction of $\bar{n}$ by an amount $v\,\Delta t$ during the time interval Δt. This shows that the plane moves with a velocity v through space. Equation (2-4-3) describes, therefore, a plane wave disturbance moving with a velocity v. The form of the function g(u) is arbitrary. Changing the sign of v in (2-4-3) is another solution of the wave equations representing a plane wave travelling in the direction of $-\bar{n}$.

Solutions of the wave equation that are of particular importance are plane waves which vary exponentially with time at any point in space. Such a plane wave can be represented in the form

$$g = A \exp\left[j(\omega t - \frac{\omega}{v}\ \bar{n} \cdot \bar{r})\right] \tag{2-4-6}$$

where ω is the frequency of oscillation and $j = \sqrt{-1}$. It is convenient to introduce the vector

$$\bar{k} = \frac{\omega}{v}\ \bar{n} \tag{2-4-7}$$

where $\bar{k}$ is known as the propagation vector. If $\bar{r}$ is advanced in an increment

$$\Delta\bar{r} = \lambda\ \bar{n} \tag{2-4-8}$$

requiring that the function g changes through a full cycle as $\bar{r}$ advances to $\bar{r} + \Delta\bar{r}$, it is found that

$$k\lambda = 2\pi \tag{2-4-9}$$

Combining (2-4-7) and (2-4-9), gives

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{v} = \omega\ \sqrt{\mu\varepsilon} \tag{2-4-10}$$

which reduces to

$$k_o = \frac{2\pi}{\lambda_o} = \frac{\omega}{c} = \omega\sqrt{\mu_o \varepsilon_o} \tag{2-4-11}$$

for free space. Note that k is the magnitude of the propagation vector k, ie,

$$k = |\bar{k}| = \frac{2\pi}{\lambda_o}\sqrt{\varepsilon_r} = k_o\sqrt{\varepsilon_r} = k_o n \tag{2-4-12}$$

It is also known as the wavenumber. For the general case in rectangular coordinates

$$\bar{k} = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z \tag{2-4-13}$$

and $$\bar{r} = x \hat{a}_x + y \hat{a}_y + z \hat{a}_z \tag{2-4-14}$$

2.5 WAVEGUIDE CONCEPTS AND TERMINOLOGY

A waveguide, in general, can be defined as any structure (usually rectangular or cylindrical) capable of guiding the flow of electromagnetic energy in a direction parallel to its axis, while substantially confining it to a region either within or adjacent to its surfaces.

A waveguide mode is an electromagnetic wave that propagates along a waveguide with a well defined phase velocity, group velocity, cross-sectional intensity distribution and polarization. Each component of its electric and magnetic field is of the form $g(x,y) \exp\left[j(\omega t - \beta z)\right]$, where the z-axis of the coordinate system is taken to coincide with the axis of the guide. These modes are referred to as the "characteristic waves" of the structures because their field vectors satisfy the wave equations in all the media that make up the guide, as well as the boundary conditions at the interfaces. Modes thus characterize the structure, in terms of its electromagnetic resonances, and depend in no way on the sources of radiation that may feed it. Each propagating mode in the interior of a waveguide is itself describable as a superposition of uniform plane waves propagating at a fixed angle with the guide axis. This superposition is so simple in the case of planar waveguides that all the basic equations governing their elementary waves may be easily derived from a direct analysis of plane waves incident and reflected from the plane interfaces that are the guiding surfaces.

The distinction between metallic and dielectric waveguides is in the mechanism responsible for confining the energy. The metallic guide does so by reflection from a good conductor at the boundary. In the dielectric waveguide, this is accomplished by total internal reflection, which is achieved by having the central dielectric made of a material of higher index of refraction than the surrounding dielectric.

3.0 ELECTROMAGNETIC WAVE PROPAGATION IN AN UNBOUNDED INHOMOGENEOUS MEDIUM

This chapter describes, in detail, the propagation of electromagnetic waves in an infinite, inhomogeneous dielectric medium with refractive index variations of the form of a perturbed parabola. The medium is assumed to be lossless, linear, isotropic, source-free and non-conducting. Cartesian coordinates have been used and the index variations are taken to be transverse to the direction of propagation. The perturbation on the parabola has the form of a polynomial of a positive integer degree. This analysis can be considered as an extension of the work done by Hashimoto |42|, who has reported an approximate perturbation method, to calculate the field distributions and the corresponding propagation constants for the first few modes of TE and TM waves using the "Related-Equation Technique" —— applying the variable transformation method suggested by Berger |43|. He |42| applied his method to an infinite medium whose refractive index has a parabolic shape with a small fourth order (quartic) perturbation on it. In this chapter the perturbed fields are expressed as infinite series in terms of Hermite functions, which are solutions for the unperturbed case. Particular examples for the case of linear, cubic and quartic perturbations have been solved, and comparisons have been made with the published results for the quartic case.

3.1 DERIVATION OF WAVE EQUATIONS FOR INHOMOGENEOUS DIELECTRICS

Consider an inhomogeneous medium (in which the permittivity ε is space dependent) having the following properties:

(a) Lossless

(b) Source-free

(c) Non-dispersive

(d) Non-conducting

(e) Non-magnetic

(f) Linear

(g) Isotropic.

The wave equations derived in section 2-3 do not hold now because $\nabla \cdot \overline{E} \neq \frac{1}{\varepsilon} \nabla \cdot \overline{D}$. Thus a new set of equations has to be derived taking the space dependence of ε into account.

Taking the curl of (2-2-2) and using A-1-2, (since $\sigma = 0$ and ε is not a constant)

$$\nabla x \nabla x \overline{H} = \nabla x \left(\varepsilon \frac{\partial}{\partial t} \overline{E}\right) = (\nabla\varepsilon) \times \frac{\partial \overline{E}}{\partial t} + \varepsilon \cdot \frac{\partial}{\partial t} (\nabla x \overline{E}) \qquad (3\text{-}1\text{-}1)$$

Using (2-2-1) in the above equation, gives

$$\nabla \times \nabla \times \overline{H} = (\nabla\varepsilon) \times \frac{\partial \overline{E}}{\partial t} - \mu\varepsilon \frac{\partial^2}{\partial t^2} \overline{H} \qquad (3\text{-}1\text{-}2)$$

Also from A-1-1,

$$\nabla \times \nabla \times \overline{H} = \nabla(\nabla \cdot \overline{H}) - \nabla^2\overline{H} = -\nabla^2\overline{H} \tag{3-1-3}$$

since $\nabla \cdot \overline{B} = \mu\nabla \cdot \overline{H} = 0$ from (2-2-4).

Thus (3-1-2) becomes

$$-\nabla^2\overline{H} = (\nabla\varepsilon) \times \frac{\partial\overline{E}}{\partial t} - \mu\varepsilon \frac{\partial^2}{\partial t^2} \overline{H} \tag{3-1-4}$$

Finally (2-2-2), with $\sigma = 0$, yields

$$\frac{\partial\overline{E}}{\partial t} = \frac{1}{\varepsilon} (\nabla \times \overline{H})$$

and using this in (3-1-4) gives

$$\nabla^2\overline{H} + \frac{1}{\varepsilon} \left[(\nabla\varepsilon) \times (\nabla \times \overline{H})\right] = \mu\varepsilon \frac{\partial^2}{\partial t^2} \overline{H} \tag{3-1-5}$$

Working on the same lines, one can determine a similar equation for $\overline{E}$. Thus from (2-2-1) (since μ is a constant, and the order of differentiation w.r.t. time and space can always be interchanged)

$$\nabla \times \nabla \times \overline{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \overline{H}) \tag{3-1-6}$$

Using (2-2-2),

$$\nabla \times \nabla \times \overline{E} = -\mu\varepsilon \frac{\partial^2}{\partial t^2} \overline{E} \qquad (3\text{-}1\text{-}7)$$

Also from A-1-1,

$$\nabla \times \nabla \times \overline{E} = \nabla(\nabla \cdot \overline{E}) - \nabla^2\overline{E} \qquad (3\text{-}1\text{-}8)$$

and from (2-2-3) (with $\rho = 0$) and (2-2-5),

$$\nabla \cdot \overline{D} = \nabla \cdot (\varepsilon\overline{E}) = 0 \qquad (3\text{-}1\text{-}9)$$

Applying vector identity A-1-3 to (3-1-9),

$$\nabla \cdot (\varepsilon\overline{E}) = (\nabla\varepsilon) \cdot \overline{E} + \varepsilon(\nabla \cdot \overline{E}) = 0 \qquad (3\text{-}1\text{-}10)$$

resulting in

$$\nabla \cdot \overline{E} = -\left[\overline{E} \cdot \frac{1}{\varepsilon} \nabla\varepsilon\right] \qquad (3\text{-}1\text{-}11)$$

Combining (3-1-7), (3-1-8) and (3-1-11), the result is

$$\nabla^2\overline{E} + \nabla \left[\overline{E} \cdot \frac{1}{\varepsilon} \nabla\varepsilon\right] = \mu\varepsilon \frac{\partial^2}{\partial t^2} \overline{E} \qquad (3\text{-}1\text{-}12)$$

Equations (3-1-5) and (3-1-12) are the wave equations for an inhomogeneous medium in which ε is space dependent. Equations of a similar form are found in acoustics and quantum mechanics. It is not surprising that the solution of these equations has received a large amount of investigation. However there is no solution when ε is an arbitrary function, and only special forms of ε yield solutions in terms of known functions.

3.2 INFINITE SQUARE-LAW MEDIA

A medium with an index of refraction 'n' having the form of a parabola (see Fig. 1).

$$n^2(x) = n_0^2 \left[1 - \delta_2 \left(\frac{x}{a}\right)^2\right] \tag{3-2-1}$$

is called a square-law medium. The square-law medium is always a first approximation to a more complicated symmetrical index distribution. If an arbitrary symmetrical index distribution is expanded in a Taylor Series, one may obtain terms higher than the second order. But in many applications, the Taylor series expansion converges rapidly, so that the first two terms are a reasonable first order approximation. However for exact analysis, some other terms must be taken into consideration.

Optical waveguides using such media are becoming of practical importance as methods of producing such media become perfected. Most waveguides, presently available, have a graded index of refraction closely resembling a square-law medium. In the present section, the

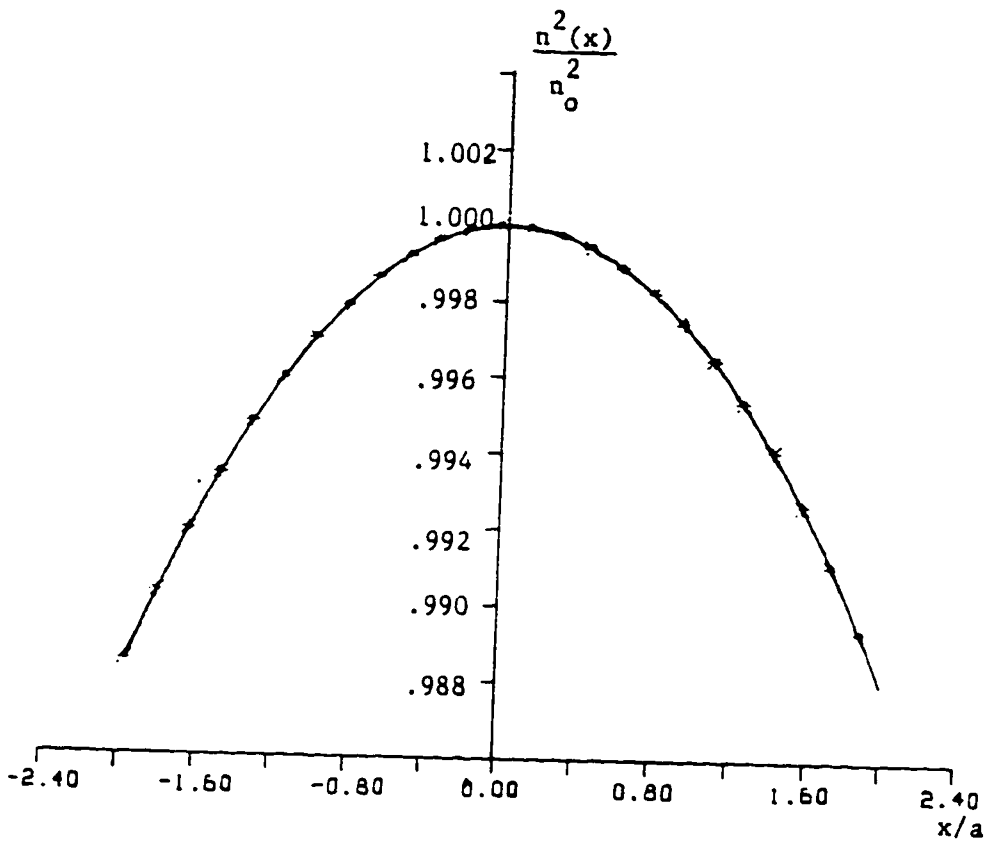


Figure 1. Parabolic refractive-index profile for an infinite medium (δ_2=0.003)

propagation characteristics of an infinite medium described by Eqn. (3-2-1) will be discussed.

For a general inhomogeneous, nonabsorbing, isotropic, linear and non-conducting dielectric medium, characterized by a dielectric constant variation $\varepsilon = \varepsilon(x,y,z)$, the wave Eqns. (3-1-5) and (3-1-12) for H_x, H_y and H_z, and E_x, E_y and E_z, respectively, are coupled. However it can be shown that for ε depending only on one coordinate (say x, as in (3-2-1)), the equations for E_x and H_x get decoupled and are of the form (assuming harmonic time variations)

$$\nabla^2 H_x + \omega^2 \mu\varepsilon\, H_x = 0 \qquad (3\text{-}2\text{-}2)$$

$$\nabla^2 E_x + \frac{d}{dx}\left[\frac{1}{\varepsilon} \cdot \frac{d\varepsilon}{dx} \cdot E_x\right] + \omega^2 \mu\varepsilon\, E_x = 0 \qquad (3\text{-}2\text{-}3)$$

Thus one can choose either $E_x = 0$ or $H_x = 0$ and express all the field components in terms of H_x and E_x |46| but in general, both E_x and H_x will exist. If the y and z dependence of the fields is of the form $\exp\left[-j\,(\gamma y + \beta z)\right]$, Eqns. (3-2-2) and (3-2-3) may be written as

$$\frac{d^2}{dx^2}\, H_x + (\omega^2\mu\varepsilon - \gamma^2 - \beta^2)\, H_x = 0 \qquad (3\text{-}2\text{-}4)$$

and

$$\frac{d^2}{dx^2}\, E_x + \frac{d}{dx}\left(\frac{1}{\varepsilon} \cdot \frac{d\varepsilon}{dx} \cdot E_x\right) + (\omega^2\mu\varepsilon - \gamma^2 - \beta^2)\, E_x = 0 \qquad (3\text{-}2\text{-}5)$$

The following cases are of special interest and can be considered without loss of generality.

Case 1: If E_x is known and H_x is assumed to be zero, whereupon Maxwell's Eqns. (2-2-1) to (2-2-4) give the other field components as

$$\left.\begin{aligned}
E_z &= \frac{-j\beta}{\varepsilon} \cdot \frac{1}{\beta^2 + \gamma^2} \cdot \frac{\partial}{\partial x} (\varepsilon E_x) \\
E_y &= \frac{-j\gamma}{\varepsilon} \cdot \frac{1}{\beta^2 + \gamma^2} \cdot \frac{\partial}{\partial x} (\varepsilon E_x) \\
H_z &= -\gamma\omega\varepsilon \cdot \frac{1}{\beta^2 + \gamma^2} \cdot E_x \\
H_y &= \beta\omega\varepsilon \cdot \frac{1}{\beta^2 + \gamma^2} \cdot E_x
\end{aligned}\right\} \qquad (3\text{-}2\text{-}6)$$

Case 2: If H_x is known and E_x is assumed to be zero, the other field components are given by

$$\left.\begin{aligned}
H_z &= -j\beta \cdot \frac{1}{\beta^2 + \gamma^2} \cdot \frac{\partial H_x}{\partial x} \\
H_y &= -j\gamma \cdot \frac{1}{\beta^2 + \gamma^2} \cdot \frac{\partial H_x}{\partial x} \\
E_z &= \gamma\omega\mu \cdot \frac{1}{\beta^2 + \gamma^2} \cdot H_x \\
E_y &= -\beta\omega\varepsilon \cdot \frac{1}{\beta^2 + \gamma^2} \cdot H_x
\end{aligned}\right\} \qquad (3\text{-}2\text{-}7)$$

From the above two sets of equations, it is obvious that if $\gamma = 0$, which implies no y dependence of the fields, case 1 represents TM modes ($H_z = 0$) and case 2 represents TE modes ($E_z = 0$). Furthermore since ε is not a constant, as is the case for a square-law medium, TEM modes cannot exist [46].

TM Modes: Since $\gamma = 0$ and E_x is known, the fourth equation in (3-2-6) gives

$$H_y = \frac{\omega\varepsilon}{\beta} E_x \quad \text{or} \quad E_x = \frac{\beta}{\omega\varepsilon} H_y \tag{3-2-8}$$

Moreover if

$$E_x = \frac{1}{\sqrt{\varepsilon}} \phi \tag{3-2-9}$$

is substituted in (3-2-5), one can eliminate the first derivative w.r.t. x, thus giving

$$\frac{d^2\phi}{dx^2} + \{\omega^2\mu\varepsilon - \frac{3}{4}(\frac{1}{\varepsilon} \cdot \frac{d\varepsilon}{dx})^2 + \frac{1}{2\varepsilon} \cdot \frac{d^2\varepsilon}{dx^2} - \beta^2\}\phi = 0 \tag{3-2-10}$$

This can be written in a compact form

$$\frac{d^2\phi}{dx^2} + \{\omega^2\mu\varepsilon - \beta^2 - \sqrt{\varepsilon_r} \cdot \frac{d^2}{dx^2} (\frac{1}{\sqrt{\varepsilon_r}})\} \phi = 0 \tag{3-2-11}$$

Finally Eqns. (3-2-8), (3-2-9) and (3-2-11) are combined to give

$$\left\{\frac{d^2}{dx^2} + k_o^2\, n^2(x) - \beta^2 - n(x)\left[\frac{1}{n(x)}\right]''\right\}\left[\frac{H_y}{n(x)}\right] = 0 \qquad (3\text{-}2\text{-}12)$$

where $\varepsilon = \varepsilon_o\, \varepsilon_r$ and $n(x) = \sqrt{\varepsilon_r}$ is the refractive index (its value at $x = 0$ is $n(0) = n_o$). Also $k_o = \omega\sqrt{\mu_o \varepsilon_o}$ is the wavenumber in free space and primes denote differentiation w.r.t. x.

<u>TE Modes</u>: In this case $\gamma = 0$ and H_x is known. Equation (3-2-7), gives

$$E_y = \frac{-\omega\mu}{\beta} H_x \quad \text{or} \quad H_x = \frac{-\beta}{\omega\mu}\, E_y \qquad (3\text{-}2\text{-}13)$$

Using this in (3-2-4), yields

$$\left\{\frac{d^2}{dx^2} + k_o^2\, n^2(x) - \beta^2\right\}\, E_y = 0 \qquad (3\text{-}2\text{-}14)$$

Equations (3-2-12) and (3-2-14) represent the cases of TM and TE modes respectively. Starting with any one of these and using either (3-2-6) or (3-2-7) can give the complete picture of either TM or TE modes. The problem now is to solve these two equations.

It can be noted that by making suitable substitutions |42|, both of the above mentioned equations can be reduced to the same form i.e.,

$$F''(x) + k^2\, [b - X(x)] \cdot F(x) = 0 \qquad (3\text{-}2\text{-}15)$$

where

$k = k_o n_o$ = wave number at the z axis

$$F(x) = \begin{cases} \dfrac{H_y}{n(x)} & \text{for TM waves} \\ \\ E_y & \text{for TE waves} \end{cases} \qquad (3\text{-}2\text{-}16)$$

$$b = \begin{cases} 1 - \left(\dfrac{\beta}{k}\right)^2 - \dfrac{n_o}{k^2} \cdot \left[\dfrac{1}{n(x)}\right]''_{x=0} & \text{for TM waves} \\ \\ 1 - \left(\dfrac{\beta}{k}\right)^2 & \text{for TE waves} \end{cases} \qquad (3\text{-}2\text{-}17)$$

$$X(x) = \begin{cases} 1 - \left[\dfrac{n(x)}{n_o}\right]^2 + \dfrac{n(x)}{k^2}\left[\dfrac{1}{n(x)}\right]'' - \dfrac{n_o}{k^2}\left[\dfrac{1}{n(x)}\right]''_{x=0} & \text{for TM waves} \\ \\ 1 - \left[\dfrac{n(x)}{n_o}\right]^2 & \text{for TE waves} \end{cases}$$

(3-2-18)

such that $X(0) = 0$. For a square-law medium, we get $X(x) = (gx)^2$, where

$$g = \begin{cases} \dfrac{\sqrt{\delta_2}}{a} \cdot \sqrt{1 + \dfrac{4\delta_2}{(ka)^2}} & \text{for TM modes} \\ \\ \dfrac{\sqrt{\delta_2}}{a} & \text{for TE modes} \end{cases} \qquad (3\text{-}2\text{-}19)$$

The modes of the square-law medium must be guided near the axis of the structure. This means that the solution $F(x)$ of (3-2-15) must vanish when $x \longrightarrow \pm \infty$. Under these conditions, the solution of (3-2-15) is known |32|, and is given by

$$\left.\begin{aligned} F_n(x) &= \exp\left[-\frac{1}{2}\xi^2\right] \cdot H_n(\xi) \\ b_n &= \frac{g}{k}(2n+1) \; ; \; n = 0,1,2,3,\ldots\ldots \end{aligned}\right\} \qquad (3\text{-}2\text{-}20)$$

where

$$\xi = (\sqrt{gk}\,) \cdot x \; ,$$

b_n is the nth eigenvalue related to Eqn. (3-2-15), and

H_n is the Hermite polynomial of nth degree.

Thus each value of n describes a mode with a unique propagation constant and a field distribution.

3.3 INFINITE MEDIUM WITH PERTURBED-PARABOLIC REFRACTIVE INDEX

Consider a refractive index of the form

$$n^2(x) = n_o^2\left[1 - \delta_2\left(\frac{x}{a}\right)^2 - \delta_p\left(\frac{x}{a}\right)^p\right] \qquad (3\text{-}3\text{-}1)$$

Where δ_p is a constant such that the third term in the brackets is very small compared to the first two, and p is any positive integer. Such an index distribution, as stated earlier, may arise as a result of the Taylor series expansion of any arbitrary refractive index distribution. Thus it is worthwhile to solve (3-2-15) for such a distribution. Exact solutions of Eqn. (3-2-15), however, are not known under such conditions. Nevertheless an approximate treatment has been reported in the literature (for p = 4 only) by Hashimoto |42|.

In the following paragraphs, a method based on the theory developed by |47| is described. The assumption is that, since the third term in Eqn. (3-3-1) is small, it can be treated as a perturbation on the refractive index of (3-2-1). Thus the resulting field distributions and the corresponding propagation constants will be close to the unperturbed case (of a parabolic index). For the unperturbed case, it is known from Eqn. (3-2-20), that the solution is

$$F_n(x) = \exp\left[-\frac{1}{2}\xi^2\right] \cdot H_n(\xi)$$

$$b_n = \frac{g}{k}(2n+1) \; ; \; n = 0,1,2,3,\ldots..$$

Since $F_n(x)$ form a complete orthogonal set, we anticipate a series solution of the form

$$F_r(x) = \sum_{n=0}^{\infty} A_{rn} \cdot F_n(x) \tag{3-3-2}$$

where $F_r(x)$ is the perturbed field for any mode r (r = 0,1,2,3,.....), in terms of the unperturbed modes n.

$$\chi(x) = (gx)^2 + \gamma(gx)^p; \quad \gamma = \delta_p \cdot (\delta_2)^{-p/2} \tag{3-3-3}$$

for any positive integer p, and

$$\left.\begin{array}{ll} g \simeq \dfrac{\sqrt{\delta_2}}{a} \cdot \sqrt{1 + \dfrac{6\gamma + 4}{(ka)^2} \cdot \delta_2} & \text{for TM modes} \\ g = \dfrac{\sqrt{\delta_2}}{a} & \text{for TE modes} \end{array}\right\} \tag{3-3-4}$$

Equation (3-2-15) now becomes*

$$F_r''(x) + k^2[b_r - \chi(x)] \cdot F_r(x) = 0 \qquad (3\text{-}3\text{-}5)$$

Using (3-3-2) in (3-3-5)

$$\sum_{n=0}^{\infty} A_{rn} F_n''(x) + k^2|b_r-(gx)^2-\gamma(gx)^p| \cdot \sum_{n=0}^{\infty} A_{rn} F_n(x) = 0 \qquad (3\text{-}3\text{-}6)$$

Using the value of $F_n''(x)$ from (3-2-15) in (3-3-6), and simplifying

$$\sum_{n=0}^{\infty} A_{rn} \cdot \{(b_r - b_n) - \gamma(gx)^p\} \cdot F_n(x) = 0 \qquad (3\text{-}3\text{-}7)$$

Multiplying (3-3-7) by $F_m(x)$, and integrating from $-\infty$ to $+\infty$,

$$\sum_{n=0}^{\infty} A_{rn} \cdot \int_{-\infty}^{\infty}(b_r-b_n) \cdot F_m(x) \cdot F_n(x) \cdot dx - \gamma \int_{-\infty}^{\infty}(gx)^p \cdot F_m(x) \cdot F_n(x) \cdot dx\} = 0$$
$$(3\text{-}3\text{-}8)$$

Since $\xi = (\sqrt{kg}) \cdot x$, $dx = \frac{d\xi}{\sqrt{kg}}$, the above equation gives

$$\sum_{n=0}^{\infty} A_{rn} \cdot \{(b_r-b_n) \cdot \int_{-\infty}^{\infty} e^{-\xi^2} \cdot H_m(\xi) \cdot H_n(\xi) \cdot d\xi - \gamma \int_{-\infty}^{\infty} (gx)^p \cdot e^{-\xi^2} H_m(\xi)\ H_n(\xi) \cdot d\xi\} = 0$$
$$(3\text{-}3\text{-}9)$$

The orthogonality condition for Hermite polynomials is

$$\int_{-\infty}^{\infty} e^{-\xi^2} \cdot H_m(\xi) \cdot H_n(\xi) \cdot d\xi = 2^n\, n! \sqrt{\pi}\, \delta_{mn} \qquad (3\text{-}3\text{-}10)$$

*A similar equation arises in the case of the Perturbed Harmonic Oscillator in Quantum Mechanics |55,57|.

where δ_{mn} is the Kronecker delta, defined as

$$\delta_{mn} = \begin{Bmatrix} 1 & \text{when } m = n \\ 0 & \text{when } m \neq n \end{Bmatrix}$$

If we let

$$\lambda_{mn} = (g)^p \cdot (kg)^{-p/2} \cdot \int_{-\infty}^{\infty} \xi^p \cdot e^{-\xi^2} \cdot H_m(\xi) \cdot H_n(\xi) \cdot d\xi \qquad (3\text{-}3\text{-}11)$$

where λ_{mn} can be evaluated in closed form |48|, as long as p is a positive integer, Eqn. (3-3-9) reduces to

$$\sum_{n=0}^{\infty} A_{rn} \cdot \left[(b_r - b_n) 2^n n! \sqrt{\pi}\, \delta_{mn} - \gamma\, \lambda_{mn}\right] = 0 \qquad (3\text{-}3\text{-}12)$$

for m = 0,1,2,3,..... and any specific value of r.

The above equation represents a system of 'm' algebraic equations in 'n' variables, where m and n both run from zero to infinity. It can also be written in an expanded form as in Eqn. (3-3-13), where $D_n = 2^n\, n! \sqrt{\pi}$ has been used.

$$\begin{bmatrix} (b_r-b_0)D_0-\gamma\lambda_{00} & -\gamma\lambda_{01} & -\gamma\lambda_{02} & \text{---} & -\gamma\lambda_{0N} & \text{---} \\ -\gamma\lambda_{10} & (b_r-b_1)D_1-\gamma\lambda_{11} & -\gamma\lambda_{12} & \text{---} & -\gamma\lambda_{1N} & \text{---} \\ -\gamma\lambda_{20} & -\gamma\lambda_{21} & (b_r-b_2)D_2-\gamma\lambda_{22} & \text{---} & -\gamma\lambda_{2N} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ -\gamma\lambda_{N0} & -\gamma\lambda_{N1} & -\gamma\lambda_{N2} & \text{---} & (b_r-b_N)D_N-\gamma\lambda_{NN} & \text{---} \end{bmatrix} \begin{bmatrix} A_{r0} \\ A_{r1} \\ A_{r2} \\ \text{---} \\ A_{rN} \end{bmatrix} = 0 \qquad (3\text{-}3\text{-}13)$$

Since the third term in Eqn. (3-3-1) is small and is being considered as a perturbation on the parabola of Eqn. (3-2-1), the terms containing higher powers of δ_p in Eqn. (3-3-13) are negligible. Hence an infinite matrix is not necessary. This allows the truncation of the series at some suitable value $n = N$, and the number of columns in the matrix of coefficients in Eqn. (3-3-13) becomes equal to N + 1. Since N + 1 equations are sufficient to compute N + 1 unknowns (i.e., A_{r0}, A_{r1}, A_{r2}, ... , A_{rN}), the maximum value of m is also taken to be N. Equation (3-3-13) is now reduced to a system of N + 1 equations in N + 1 unknowns, and can be rearranged as follows (with b_r as a parameter):

$$AV = b_r \, V \tag{3-3-14}$$

where A is the matrix of coefficients of the system (3-3-14), having N + 1 rows and N + 1 columns, given by

$$A = \begin{bmatrix} b_0 + \frac{\gamma\lambda_{00}}{D_0} & \frac{\gamma\lambda_{01}}{D_1} & \frac{\gamma\lambda_{02}}{D_2} & ---- & \frac{\gamma\lambda_{0N}}{D_N} \\ \frac{\gamma\lambda_{10}}{D_0} & b_1 + \frac{\gamma\lambda_{11}}{D_1} & \frac{\gamma\lambda_{12}}{D_2} & ---- & \frac{\gamma\lambda_{1N}}{D_N} \\ \frac{\gamma\lambda_{20}}{D_0} & \frac{\gamma\lambda_{21}}{D_1} & b_2 + \frac{\gamma\lambda_{22}}{D_2} & ---- & \frac{\gamma\lambda_{2N}}{D_N} \\ ----- & ----- & ----- & ---- & ----- \\ \frac{\gamma\lambda_{N0}}{D_0} & \frac{\gamma\lambda_{N1}}{D_1} & \frac{\gamma\lambda_{N2}}{D_2} & ---- & b_N + \frac{\gamma\lambda_{NN}}{D_N} \end{bmatrix} \tag{3-3-15}$$

and V is the vector of N + 1 unknowns given by

$$V = \begin{bmatrix} A_{r0} \\ A_{r1} \\ A_{r2} \\ --- \\ --- \\ --- \\ A_{rN} \end{bmatrix} \qquad (3\text{-}3\text{-}16)$$

The values of b_r correspond to the roots (N + 1 in this case) and A_{rn}'s are the corresponding eigenvectors |49,55| for the determinant of the matrix A. Knowing b_r's, the propagation constants for the perturbed modes can be calculated, and knowing A_{rn}'s , the perturbed fields can be obtained. These determine E_y or H_y, and the other components of the electric and magnetic field can then be computed using Eqns. (3-2-6) and (3-2-7).

3.4 EVALUATION OF λ_{mn}

From Eqn. (3-3-11),

$$\lambda_{mn} = g^p (kg)^{-p/2} \cdot \int_{-\infty}^{\infty} \xi^p \cdot \exp(-\xi^2) \cdot H_m(\xi) \cdot H_n(\xi) \cdot d\xi$$

It has been shown by |48| that when p is a positive integer,

$$\int_{-\infty}^{\infty} \xi^p \cdot \exp(-\xi^2) \cdot H_m(\xi) \cdot H_n(\xi) \cdot d\xi = m!\ n!\ \cdot C_{mn} \qquad (3\text{-}4\text{-}1)$$

where C_{mn} represents the coefficient of the term $z^m\ y^n$ in the expansion of I_p, which is given by

$$I_p = e^{2yz} \cdot \sum_{i=0}^{p} \left[(y+z)^i \cdot {}^pC_i \cdot \Lambda \right]$$

and

$$\Lambda = \begin{cases} \Gamma\left(\frac{p-i+1}{2}\right) & \text{when } (p-i) \text{ is even} \\ 0 & \text{when } (p-i) \text{ is odd} \end{cases} \qquad (3\text{-}4\text{-}2)$$

In the above equation, ${}^pC_i$ represents the combination of 'p' taken 'i' at a time, and $\Gamma\left(\frac{p-i+1}{2}\right)$ stands for the Gamma function whose argument is $\left(\frac{p-i+1}{2}\right)$.

Thus for any value of p,

$$\lambda_{mn} = g^p (kg)^{-p/2} \cdot m!\ n!\ C_{mn} \qquad (3\text{-}4\text{-}3)$$

Note that $\lambda_{mn} = \lambda_{nm}$

3.5 ILLUSTRATIVE EXAMPLES AND DISCUSSIONS

To demonstrate the application of the previous results, three illustrative cases, mentioned below were considered:

(a) Linear perturbation

$$n^2(x) = n_o^2 \left[1 - \delta_2\left(\frac{x}{a}\right)^2 - \delta_1\left(\frac{x}{a}\right)\right] \qquad (3\text{-}5\text{-}1)$$

(b) Cubic perturbation

$$n^2(x) = n_o^2 \left[1 - \delta_2\left(\frac{x}{a}\right)^2 - \delta_3\left(\frac{x}{a}\right)^3\right] \qquad (3\text{-}5\text{-}2)$$

(c) Quartic perturbation

$$n^2(x) = n_o^2 \left[1 - \delta_2\left(\frac{x}{a}\right)^2 - \delta_4\left(\frac{x}{a}\right)^4\right] \qquad (3\text{-}5\text{-}3)$$

For each of the above cases, the coefficients used were

$\delta_2 = 0.003$ $\qquad$ $n_o = 1.0$

$f = 1 \times 10^{12}$ Hz $\qquad$ $\lambda_o = 3 \times 10^{-4}$ m

$a = 44\lambda_o = 1.32$ cms

hence $\quad \frac{g}{k} = \frac{\sqrt{\delta_2}}{a} \cdot \frac{\lambda_o}{2\pi n_o} = \frac{\sqrt{0.003}}{1.32 \times 10^{-2}} \cdot \frac{3 \times 10^{-4}}{2\pi \times 1.0} = 2 \times 10^{-4}$

Moreover $\delta_1 = 0.0015$, $\delta_3 = 0.0025$ and $\delta_4 = 0.0045$

The selection of the above mentioned constants is totally arbitrary, as long as $\left|\delta_p\left(\frac{x}{a}\right)^p\right|$ is small, however the values of δ_2

and δ_4 are those used by |42|. The reason for this choice is to allow for a possibility of comparison between case (c) of our examples and his results.

A general computer program (listed in Appendix A-2) was written in Fortran IV, which enables us to handle any one of the above cases completely, by specifying the order of perturbation and other required constants. As a first step the coefficients C_{mn} (see Eqn. (3-4-3)) are tabulated and arranged in the form of a matrix. Since only the first few modes are of interest, the value of N is taken to be 5 thus C is a 6 x 6 matrix. Using these values of C_{mn}, λ_{mn} (Eqn. (3-4-3)) and then the matrix A (Eqn. (3-3-15)) is computed. In the next step, the eigenvalues and eigenvectors of matrix A are evaluated.* For our case, we have 6 eigenvalues and 6 eigenvectors. Finally using Eqn. (3-3-2), the perturbed field is found for each of the eigenvalues evaluated.

Important results for case (a), (b) and (c) are listed in Tables I, II and III respectively. The rows of matrix A correspond to values of m, and its columns to values of n, but in the case of the matrix representing the eigenvectors, the rows correspond to n and the columns correspond to r because the eigenvalues are listed in the sequence b_0, b_1, b_2, ... , b_5 (under each eigenvalue is its corresponding eigenvector). Table IV shows a comparison of the computed eigenvalues 'b_r' for these cases along with the unperturbed eigenvalues 'b_n' for n & r = 0,1,2,.... , 5. In Fig. 2(a) through (f) the perturbed fields 'F_r' along with the unperturbed fields 'F_n'

*The Diagonalization Method originated by Jacobi and adapted by Von Neumann is used in the SSP Subroutines [14].

TABLE I(a). Computer Results for Linear Perturbations (N=5).

MATRIX A

0.0002000	0.0001937	0.0	0.0	0.0	0.0
0.0003875	0.0006000	0.0001937	0.0	0.0	0.0
0.0	0.0007750	0.0010000	0.0001937	0.0	0.0
0.0	0.0	0.0011625	0.0014000	0.0001937	0.0
0.0	0.0	0.0	0.0015500	0.0018000	0.0001937
0.0	0.0	0.0	0.0	0.0019375	0.0022000

EIGEN VALUES OF MATRIX A I.E., ER'S

0.0001149	0.0005916	0.0009996	0.0014003	0.0018084	0.0022851

EIGENVECTORS OF MATRIX A COLUMNWISE.

0.9117417	0.3970336	0.1036337	0.0174835	0.0021213	0.0001503
−0.4003823	0.8024901	0.4285537	0.1083179	0.0176099	0.0016175
0.0906909	−0.4318078	0.7801586	0.4299703	0.1077108	0.0139171
−0.0139171	0.1077107	−0.4299704	0.7801586	0.4318076	0.0906910
0.0016174	−0.0176099	0.1083180	−0.4285538	0.8024903	0.4003817
−0.0001503	0.0021212	−0.0174835	0.1036336	−0.3970332	0.9117420

TABLE I(b). Linear Perturbations with N=4.

EIGEN VALUES OF MATRIX A I.E., BR'S

0.0001149	0.0005916	0.0010000	0.0014084	0.0018851

EIGENVECTORS OF MATRIX A COLUMNWISE.

0.9117419	0.3970411	0.1038508	0.0172556	0.0015999
-0.4003823	0.8025197	0.4288122	0.1076219	0.0139143
0.0906903	-0.4317873	0.7814538	0.4317874	0.0906904
-0.0139143	0.1076218	-0.4288124	0.8025198	0.4003820
0.0015999	-0.0172556	0.1038510	-0.3970408	0.9117419

TABLE I(c). Linear Perturbations with N=3.

EIGEN VALUES OF MATRIX A I.E., BR'S

0.0001149	0.0005920	0.0010080	0.0014851

EIGENVECTORS OF MATRIX A COLUMNWISE.

0.9117582	0.3973000	0.1032619	0.0136645
-0.4003696	0.8037511	0.4306669	0.0906335
0.0906336	-0.4306667	0.8037513	0.4003692
-0.0136646	0.1032619	-0.3972996	0.9117584

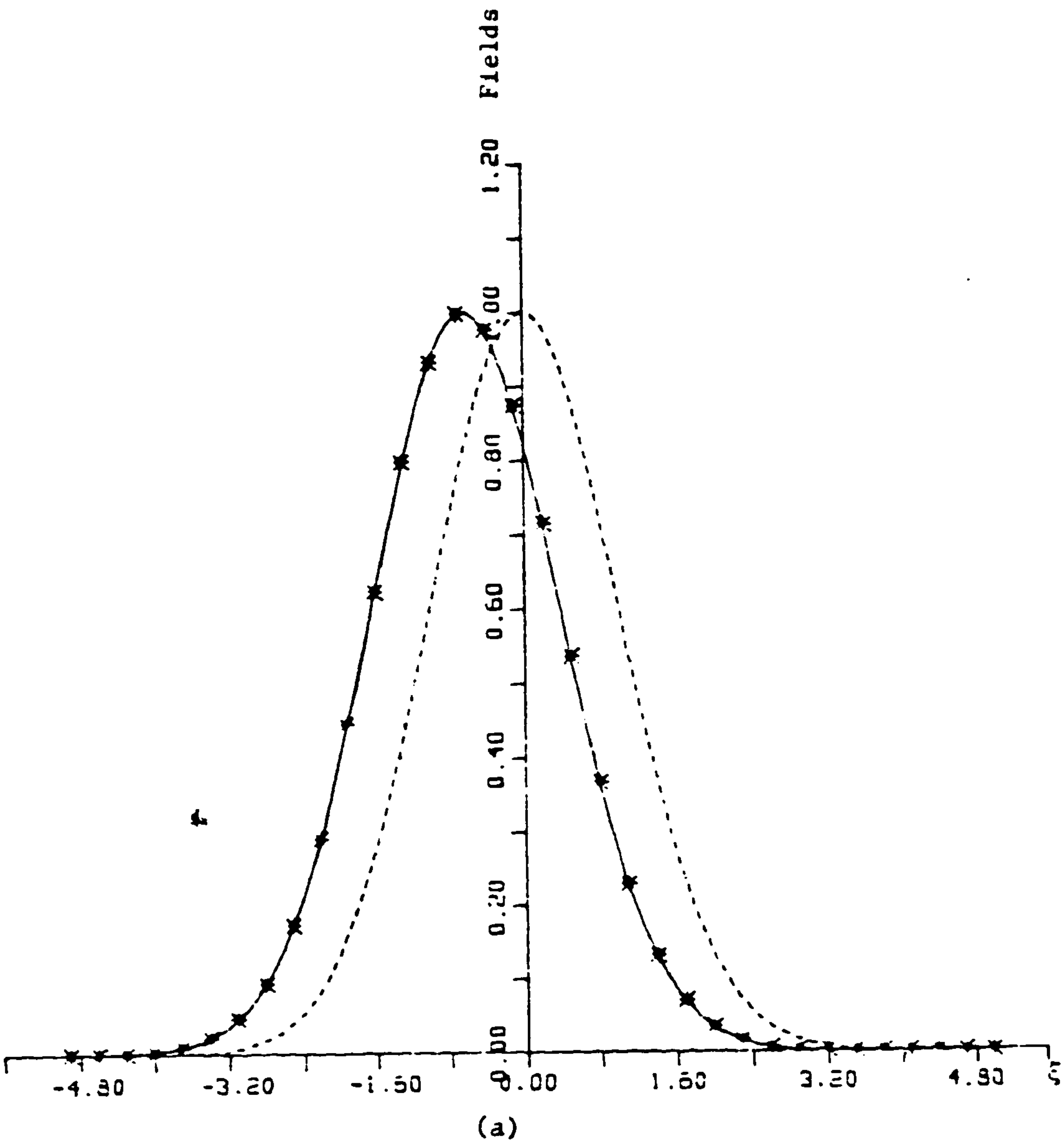
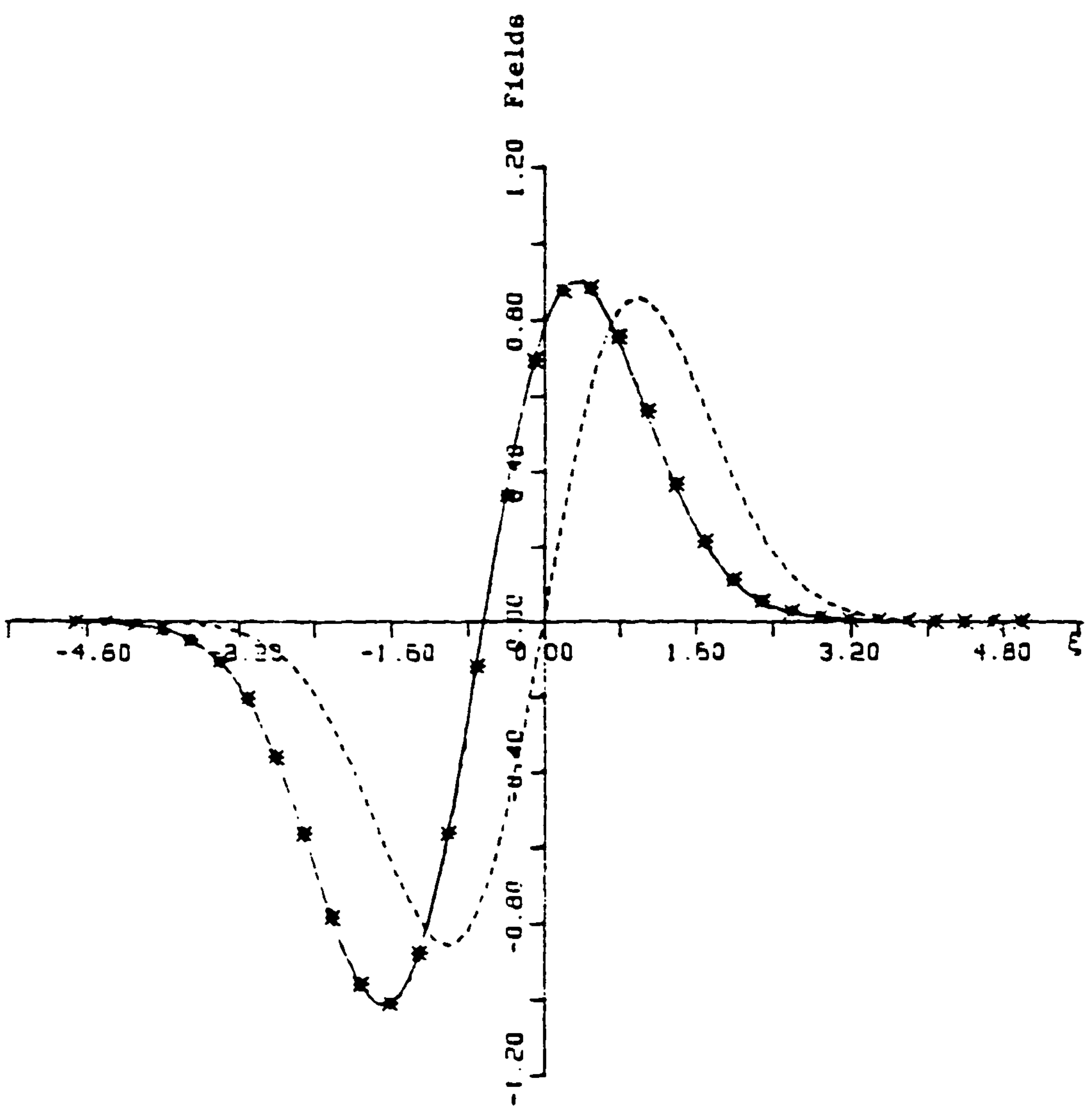


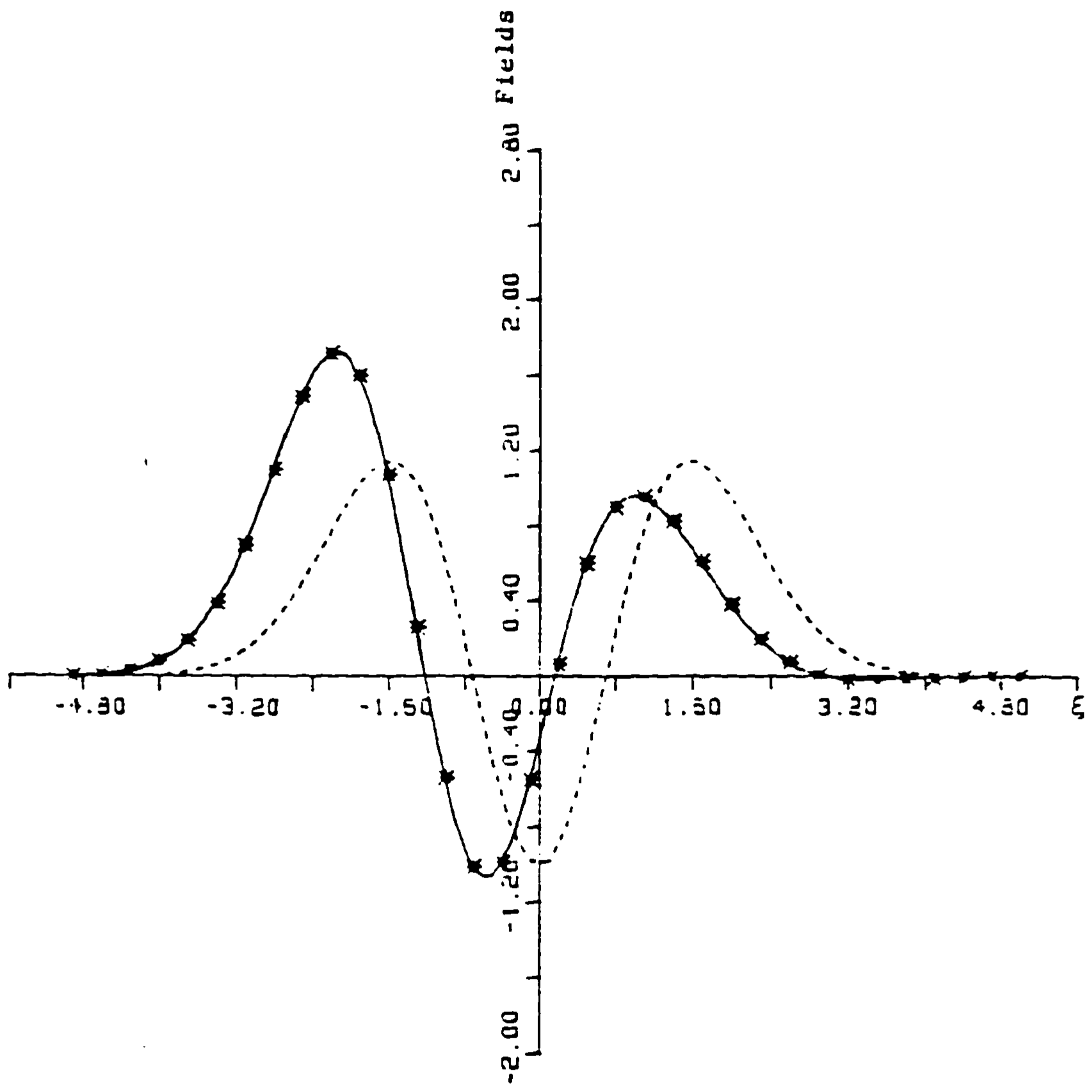
Figure 2. Field distributions for Linear Perturbation

Fundamental mode (n=0) for p=1.



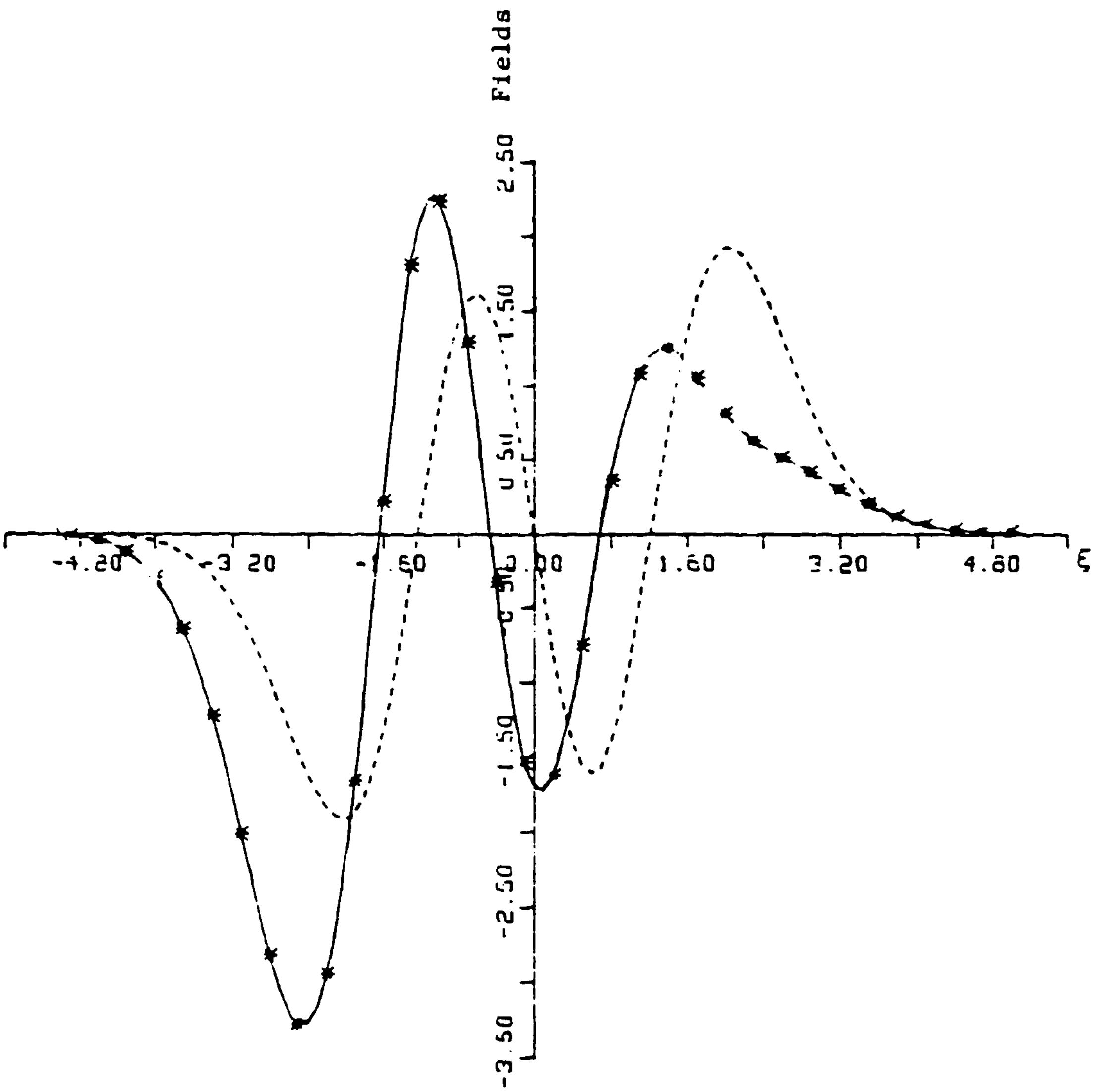
(b)

First mode (n=1) for p=1.



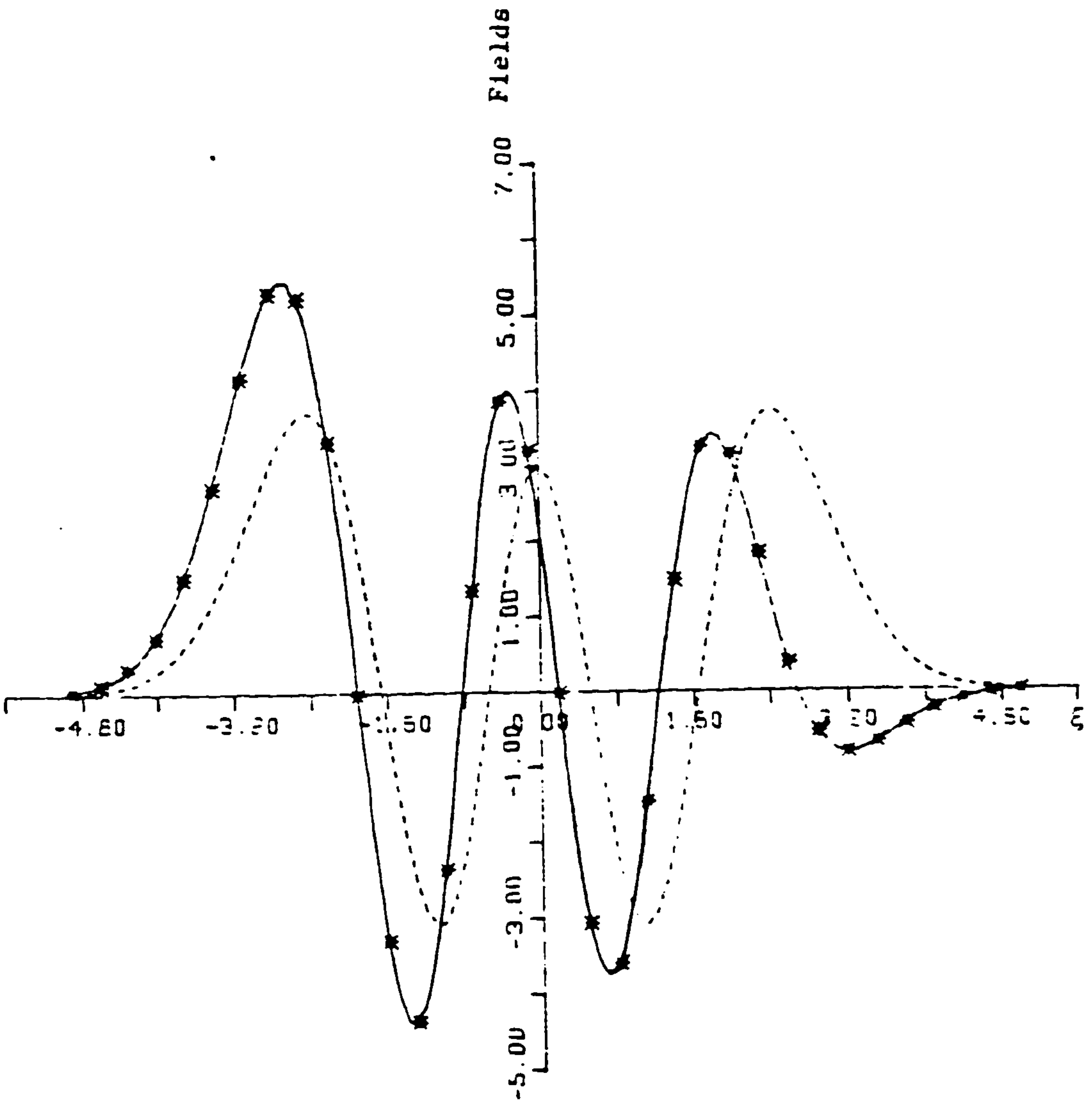
(c)

Second mode (n=2) for p=1.



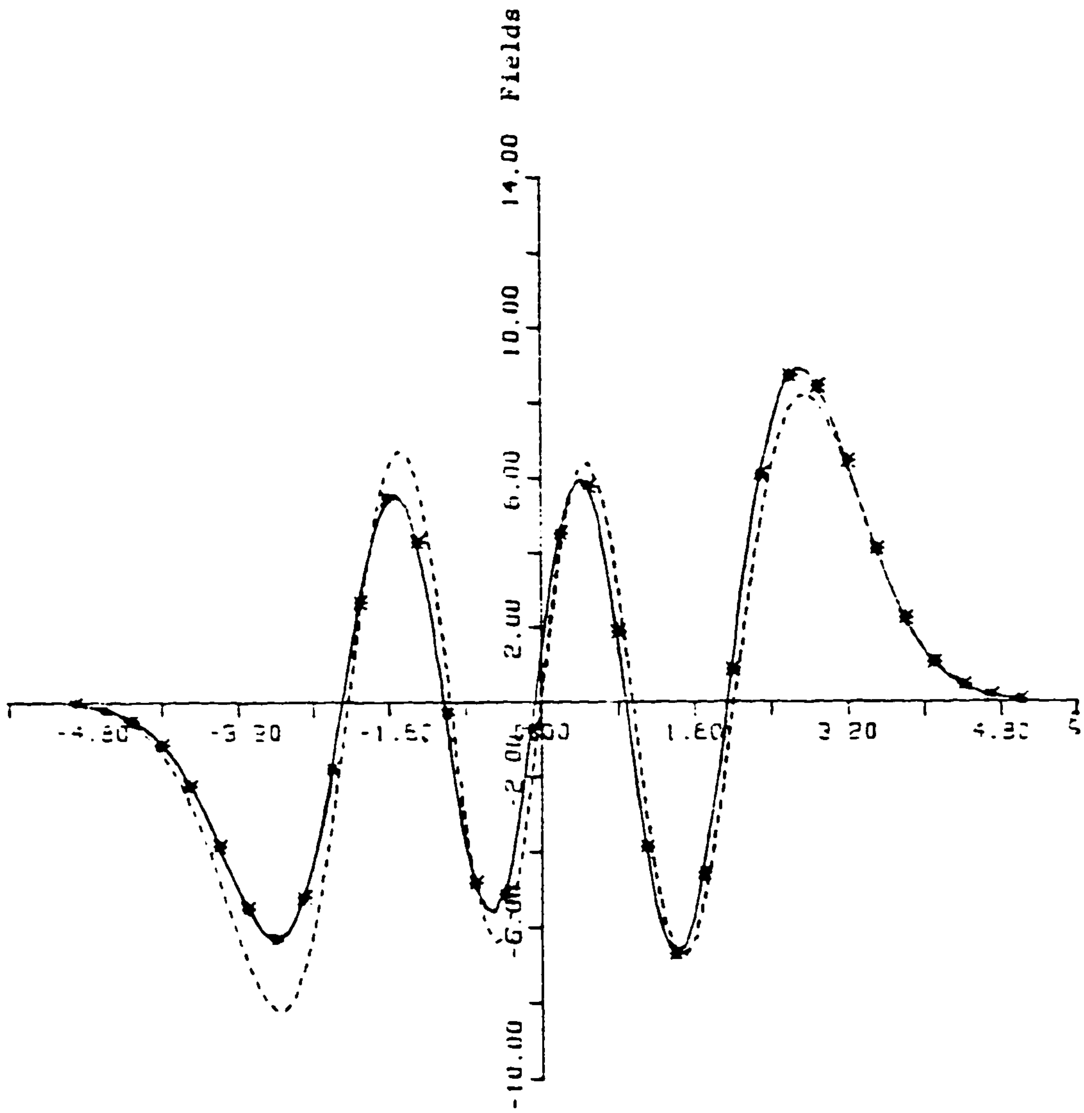
(d)

Third mode (n=3) for p=1.



(e)

Fourth mode (n=4) for p=1.



(f)

Fifth mode (n=5) for p=1.

TABLE II(a). Computer Results for Cubic Perturbations (N=5).

MATRIX A

0.0002000	0.0000318	0.0	0.0000053	0.0	0.0
0.0000636	0.0006000	0.0000636	0.0	0.0000053	0.0
0.0	0.0002546	0.0010000	0.0000955	0.0	0.0000053
0.0002546	0.0	0.0005728	0.0014000	0.0001273	0.0
0.0	0.0010182	0.0	0.0010182	0.0018000	0.0001591
0.0	0.0	0.0025456	0.0	0.0015910	0.0022000

EIGEN VALUES OF MATRIX A I.E., BR'S

0.0001974	0.0005923	0.0009872	0.0013824	0.0017828	0.0022578

EIGENVECTORS OF MATRIX A COLUMNWISE.

0.9967668	0.0800632	0.0048331	0.0045963	0.0011390	0.0001541
-0.0798869	0.9839615	0.1584507	0.0171468	0.0059168	0.0013970
0.0069319	-0.1581082	0.9586385	0.2337877	0.0353031	0.0079316
-0.0050166	0.0191849	-0.2332606	0.9218835	0.3044368	0.0514214
0.0006699	-0.0064962	0.0372727	-0.3030796	0.8892196	0.3405870
-0.0000716	0.0011644	-0.0090617	0.0574598	-0.3395926	0.9387713

TABLE II(b). Cubic Perturbations with N=4.

EIGEN VALUES OF MATRIX A I.E., BR'S

0.0001974	0.0005923	0.0009873	0.0013849	0.0018380

EIGENVECTORS OF MATRIX A COLUMNWISE.

0.9967669	0.0800629	0.0048345	0.0046211	0.0010376
-0.0798888	0.9839669	0.1584235	0.0171782	0.0058245
0.0069313	-0.1580884	0.9588631	0.2333347	0.0329695
-0.0050160	0.0191580	-0.2327996	0.9294522	0.2855361
0.0006628	-0.0063400	0.0354272	-0.2852318	0.9577824

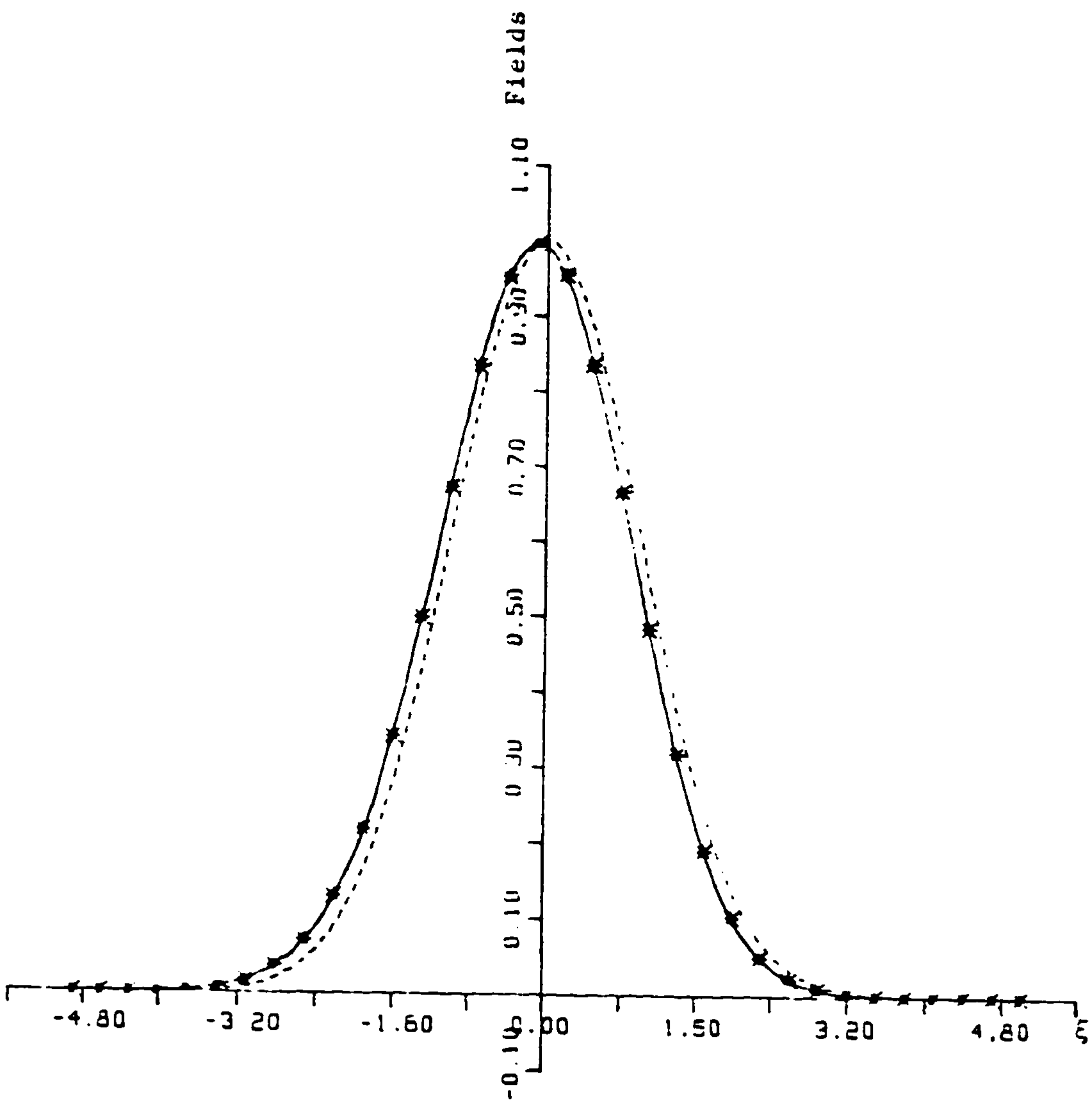
TABLE II(c). Cubic Perturbations with N=3.

EIGEN VALUES OF MATRIX A I.E., BR'S

0.0001974	0.0005924	0.0009883	0.0014219

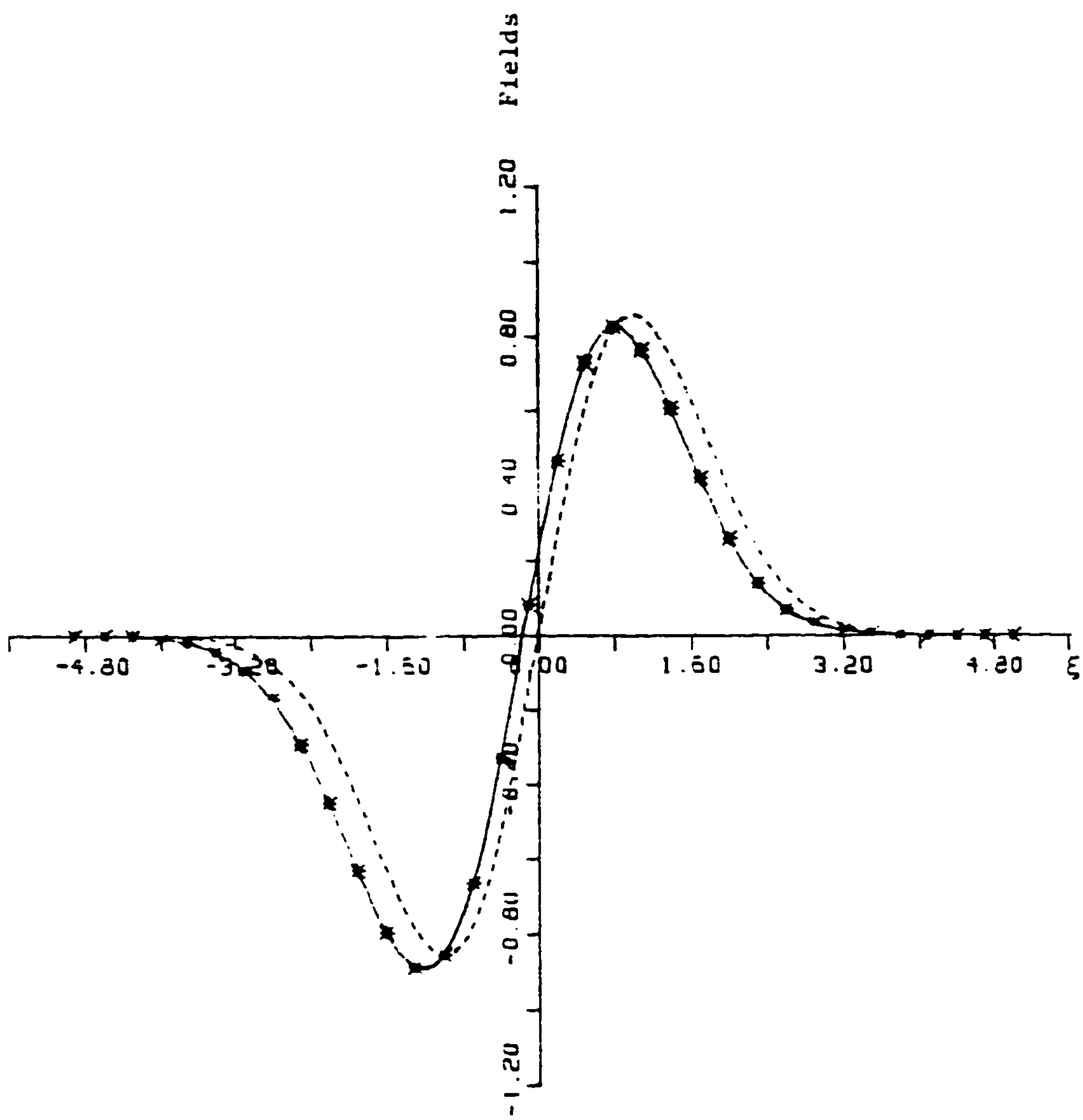
EIGENVECTORS OF MATRIX A COLUMNWISE.

0.9967683	0.0800455	0.0048783	0.0046848
-0.0798789	0.9840417	0.1580395	0.0174609
0.0069221	-0.1578792	0.9618868	0.2231591
-0.0049452	0.0181354	-0.2230998	0.9746144



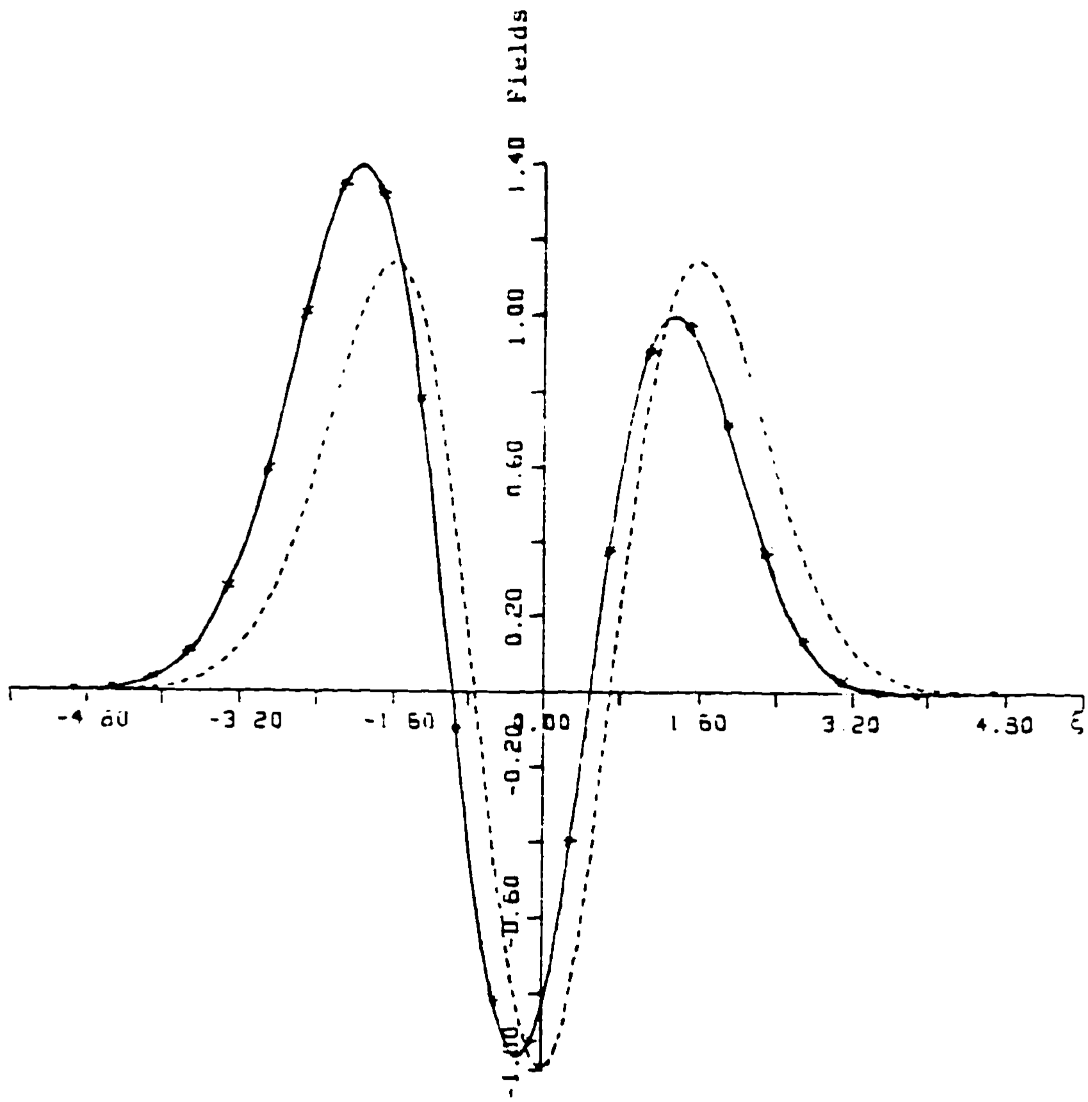
(a)

Figure 3. Field distributions for Cubic Perturbations
Fundamental mode (n=0) for p=3.



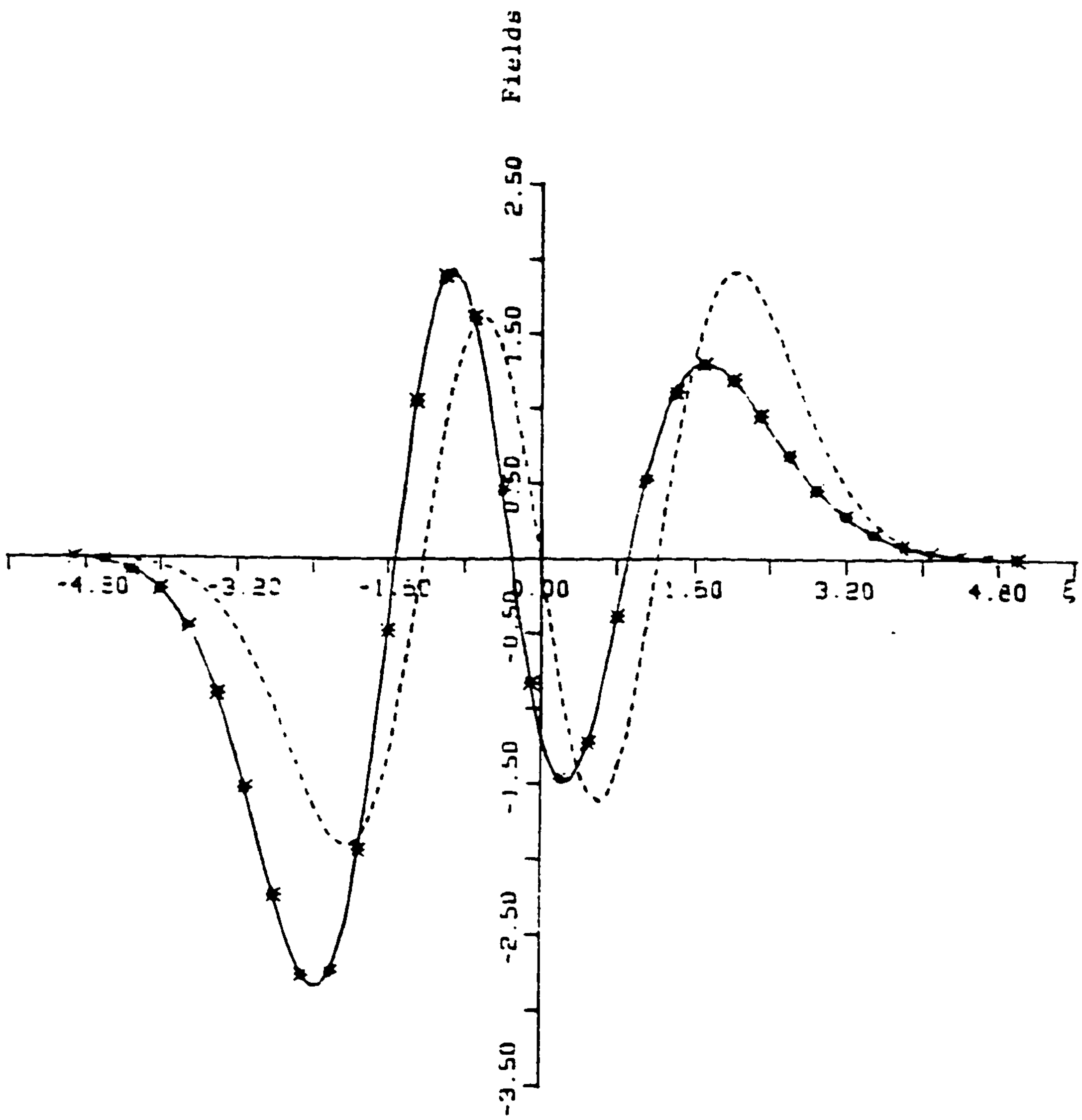
(b)

First mode (n=1) for p=3.



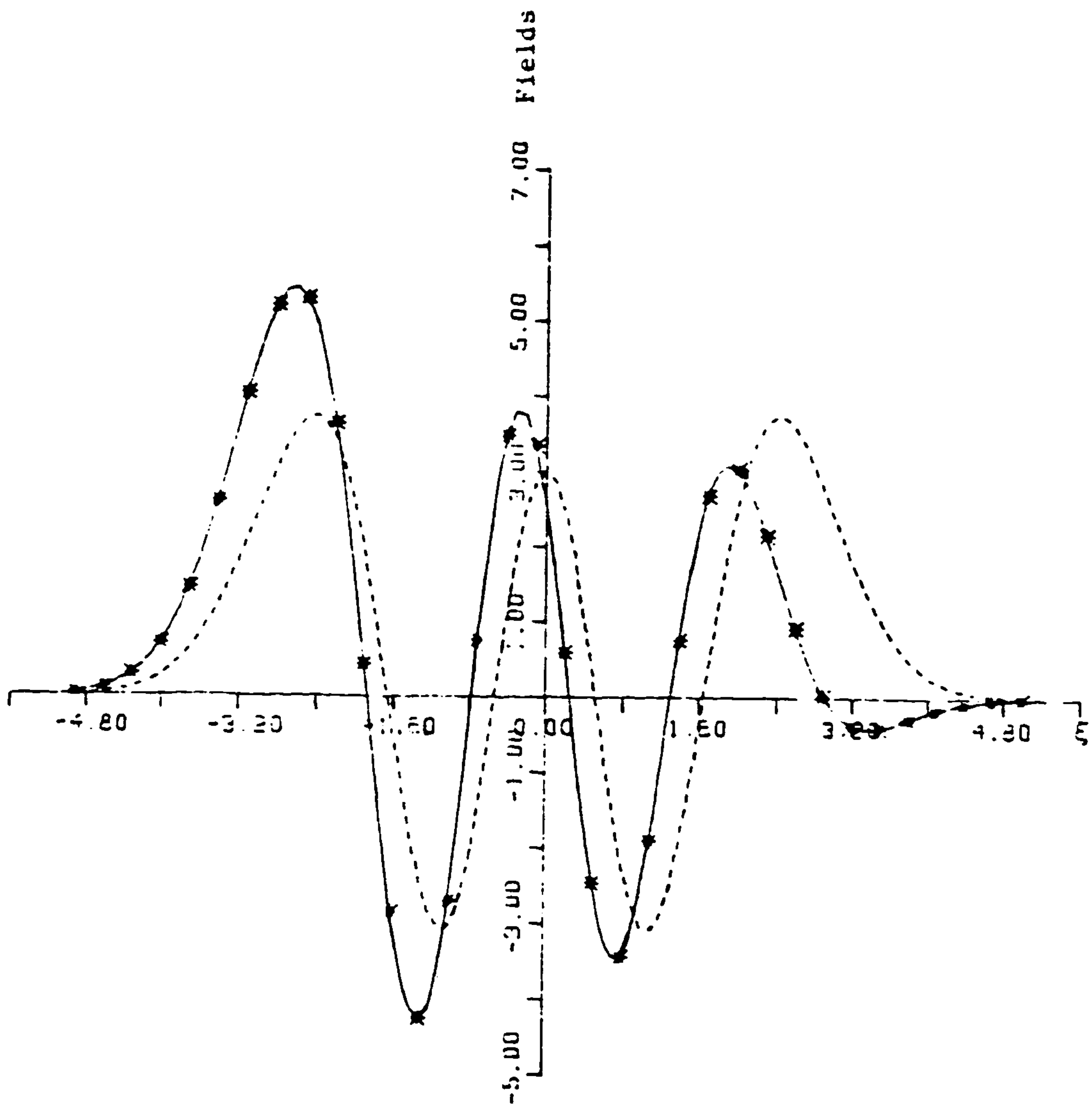
(c)

Second mode (n=2) for p=3.



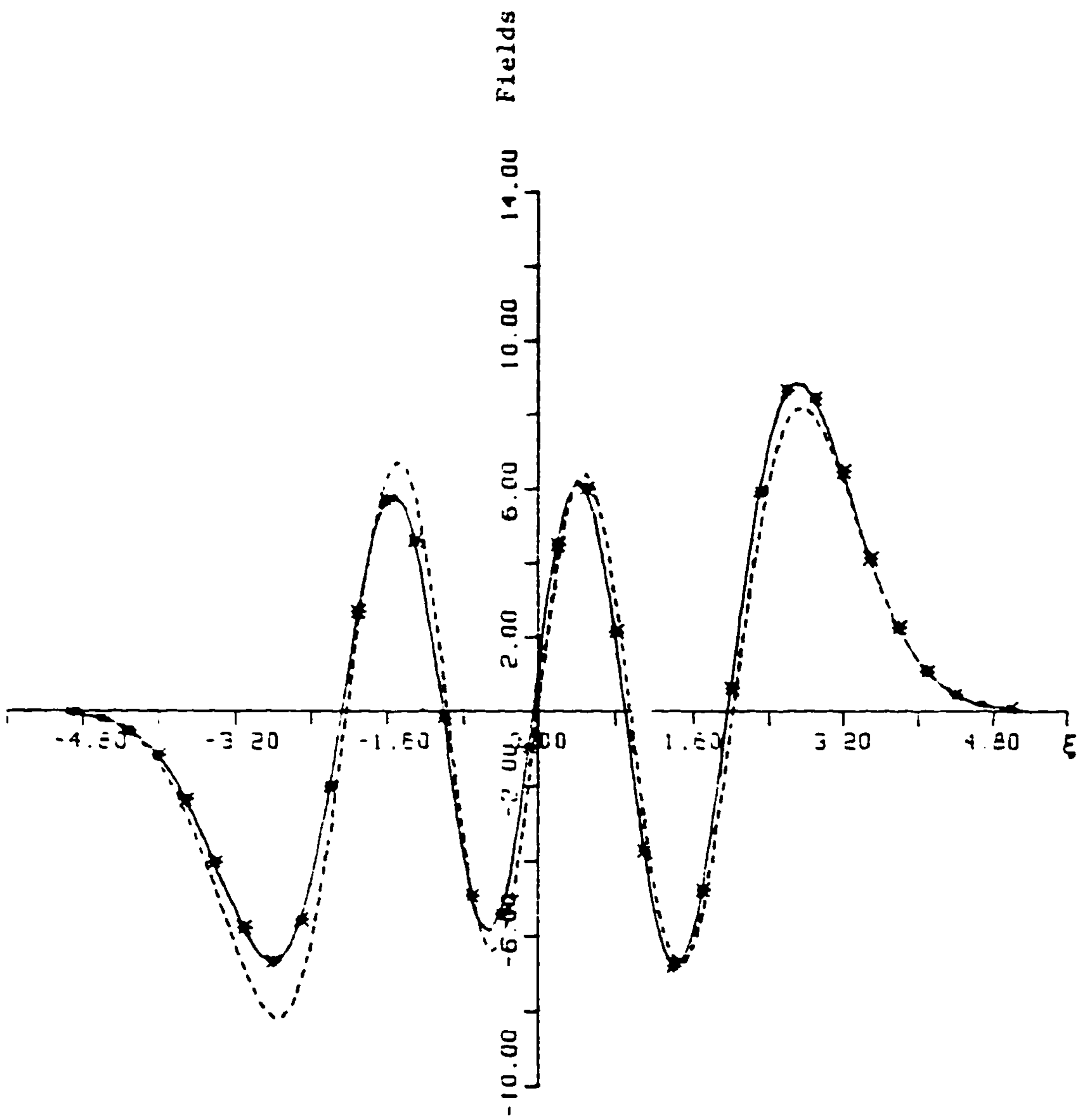
(d)

Third mode (n=3) for p=3.



(e)

F... ch mode (n=4) for p=3.



(f)

Fifth mode (n=5) for p=3.

TABLE III(a). Computer Results for Quartic Perturbations (N=5).

MATRIX A

0.0002150	0.0	0.0000150	0.0	0.0000012	0.0
0.0	0.0006750	0.0	0.0000250	0.0	0.0000012
0.0001200	0.0	0.0011950	0.0	0.0000350	0.0
0.0	0.0006000	0.0	0.0017750	0.0	0.0000450
0.0004800	0.0	0.0016800	0.0	0.0024150	0.0
0.0	0.0024000	0.0	0.0036000	0.0	0.0031150

EIGEN VALUES OF MATRIX A I.E., ER'S

0.0002148	0.0006744	0.0011942	0.0017741	0.0024160	0.0031165

EIGENVECTORS OF MATRIX A COLUMNWISE.

0.9998831	0.0	0.0152735	0.0	0.0007630	0.0
0.0	0.9997422	0.0	0.0226899	0.0	0.0008551
-0.0152891	0.0	0.9994722	0.0	0.0286625	0.0
0.0	-0.0227058	0.0	0.9991794	0.0	0.0335412
-0.0003248	0.0	-0.0286706	0.0	0.9995668	0.0
0.0	-0.0000934	0.0	-0.0335520	0.0	0.9994370

TABLE III(b). Quartic Perturbations with N=4.

EIGEN VALUES OF MATRIX A I.E., BR'S

0.0002148	0.0006744	0.0011942	0.0017756	0.0024160

EIGENVECTORS OF MATRIX A COLUMNWISE.

0.9998831	0.0	0.0152735	0.0	0.0007630
0.0	0.9997422	0.0	0.0227096	0.0
-0.0152891	0.0	0.9994722	0.0	0.0286624
0.0	-0.0227096	0.0	0.9997422	0.0
-0.0003248	0.0	-0.0286708	0.0	0.9995888

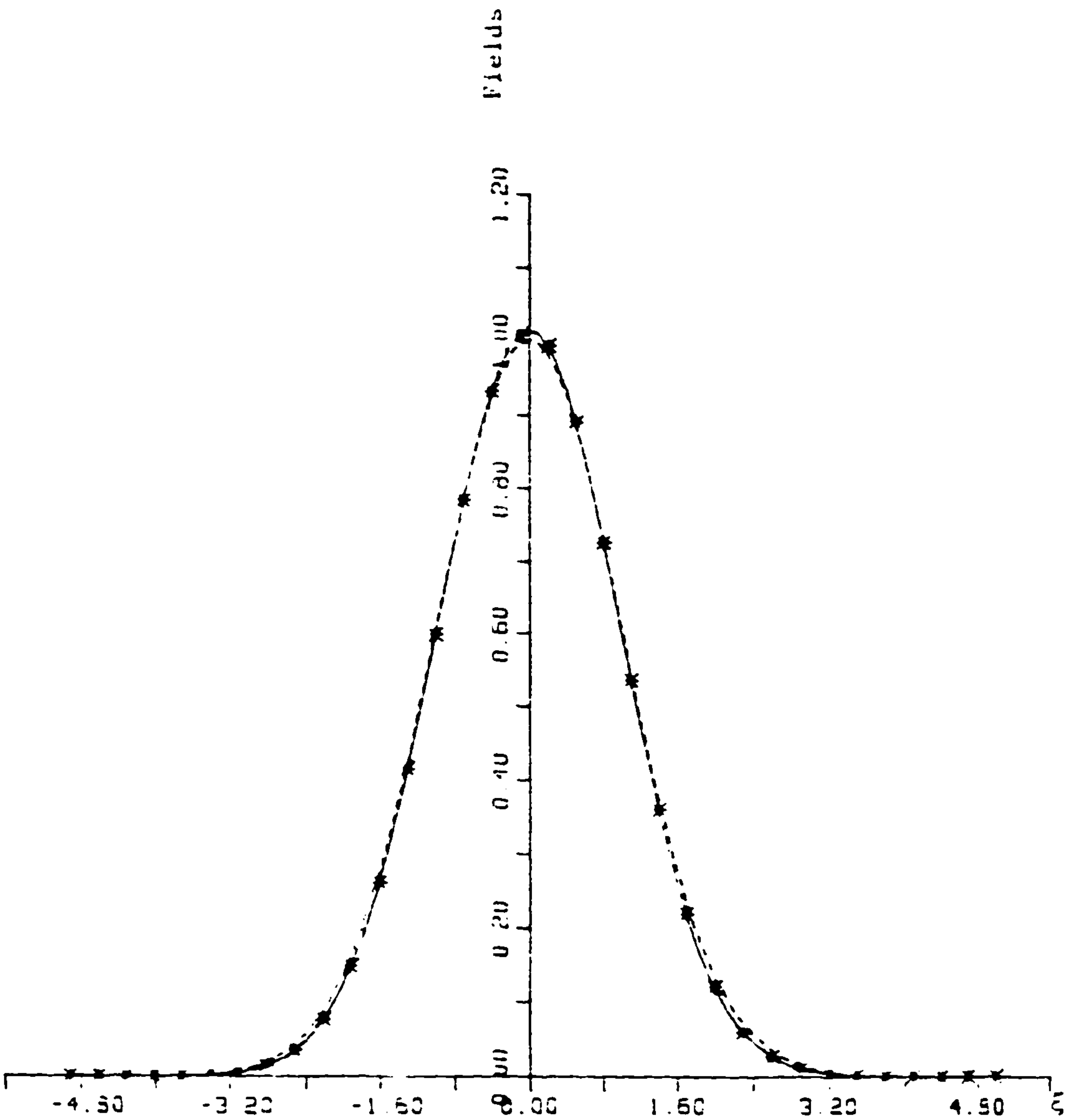
TABLE III(c). Quartic Perturbations with N=3.

EIGEN VALUES OF MATRIX A I.E., BR'S

0.0002148	0.0006744	0.0011952	0.0017756

EIGENVECTORS OF MATRIX A COLUMNWISE.

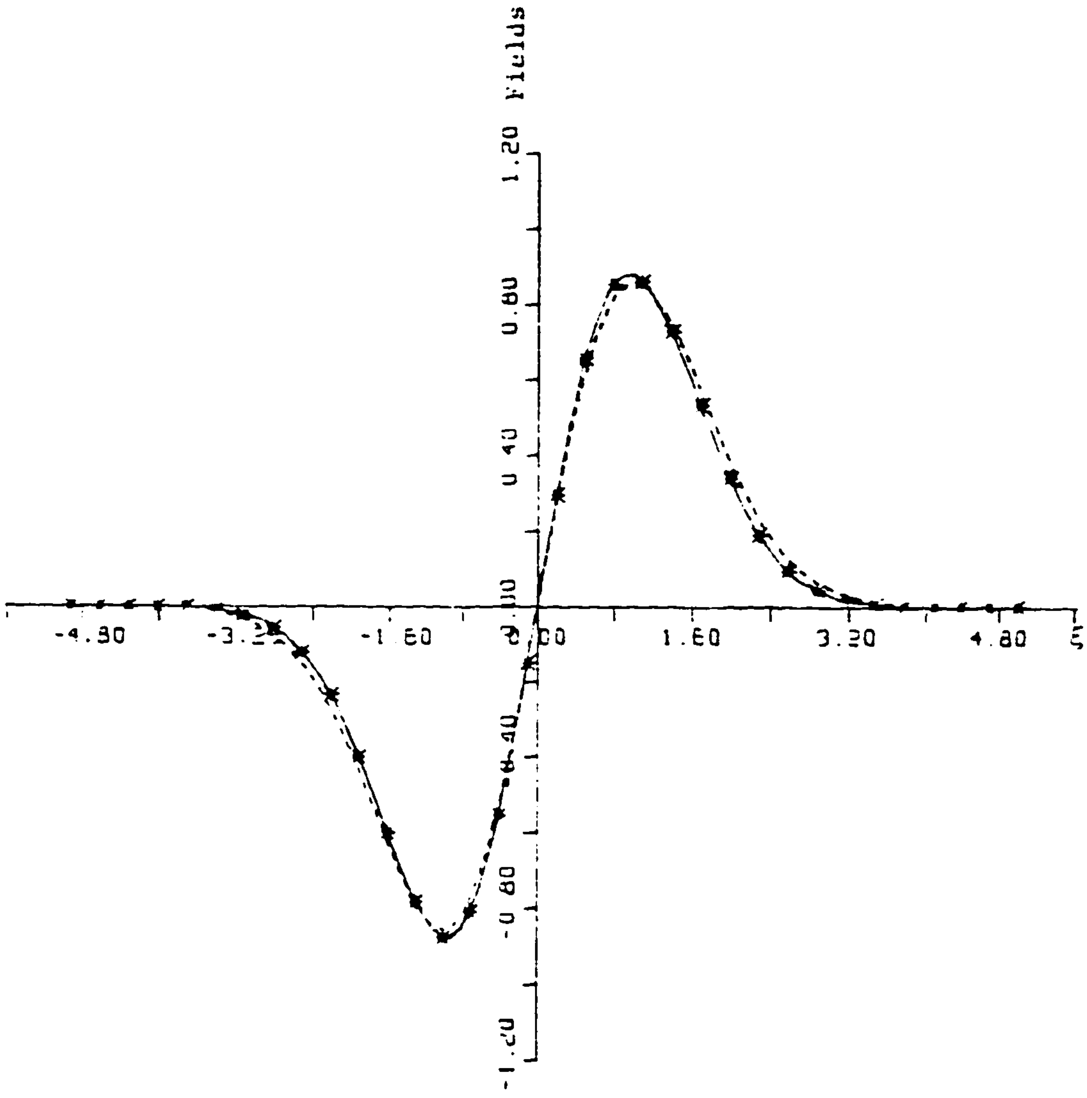
0.9998829	0.0	0.0153007	0.0
0.0	0.9997422	0.0	0.0227096
-0.0153007	0.0	0.9998829	0.0
0.0	-0.0227096	0.0	0.9997422



(a)

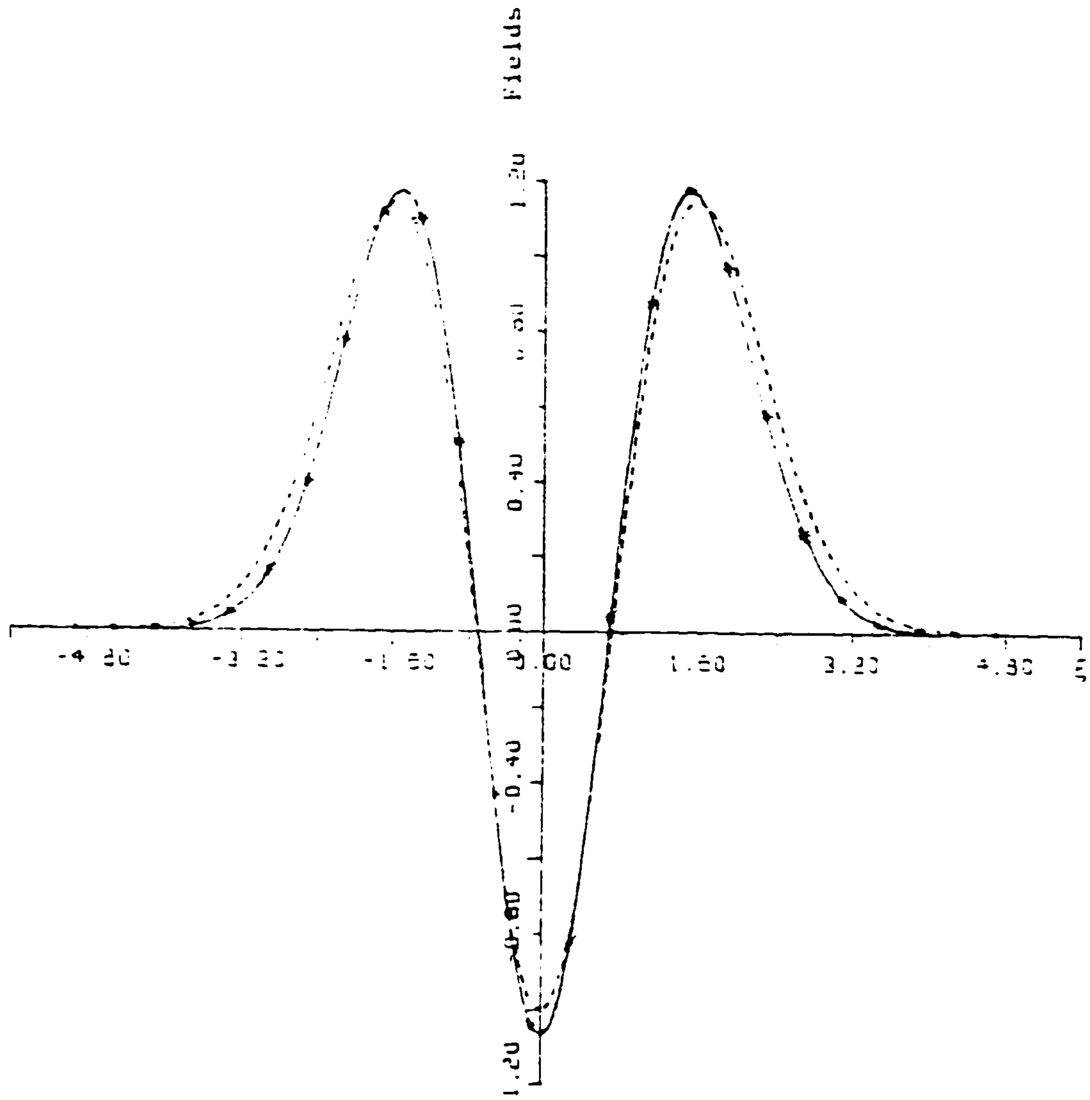
Figure 4. Field distributions for Quartic Perturbation.

Fundamental Mode (n=0) for p=4.



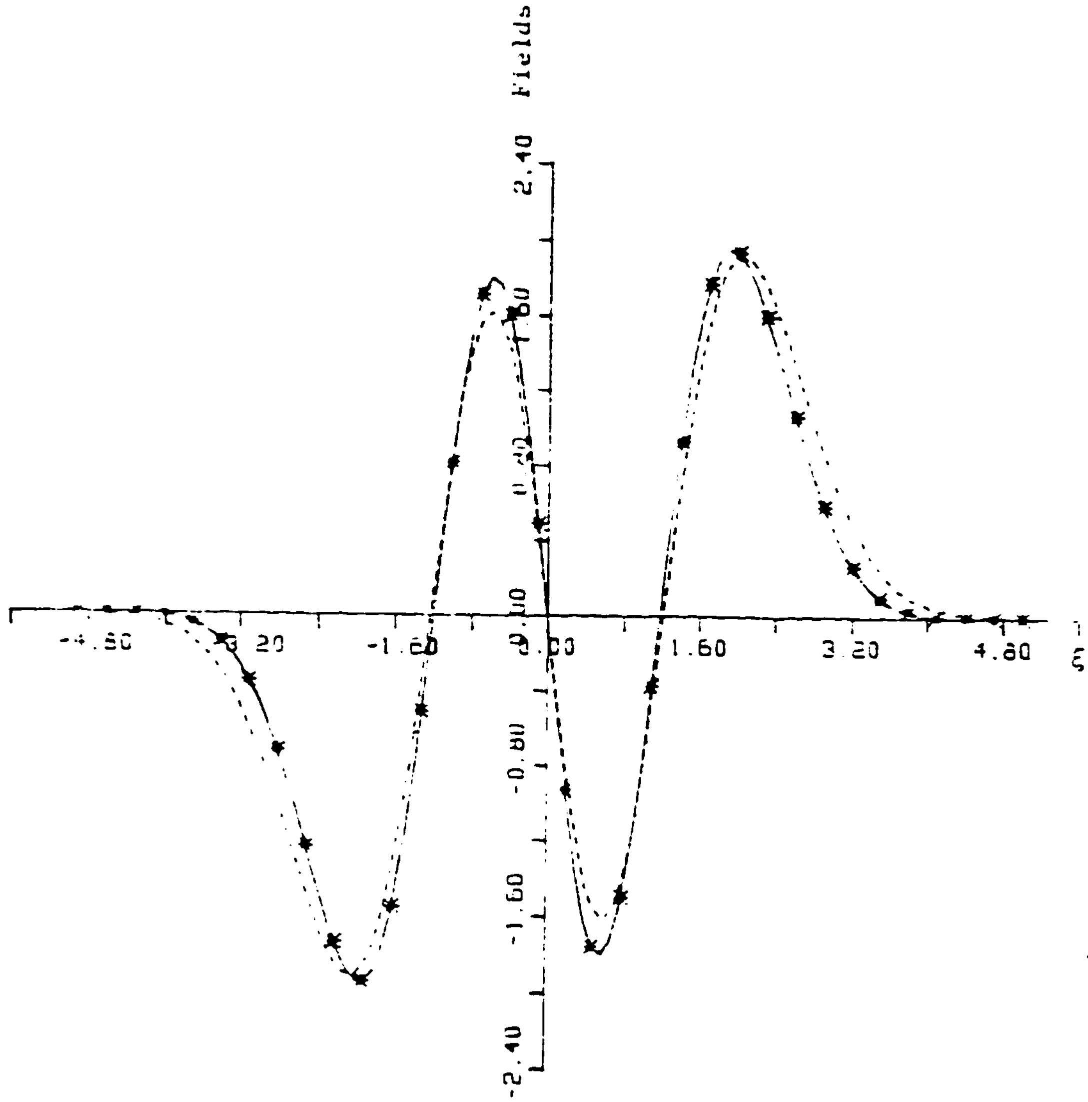
(b)

First mode (n=1) for p=4.



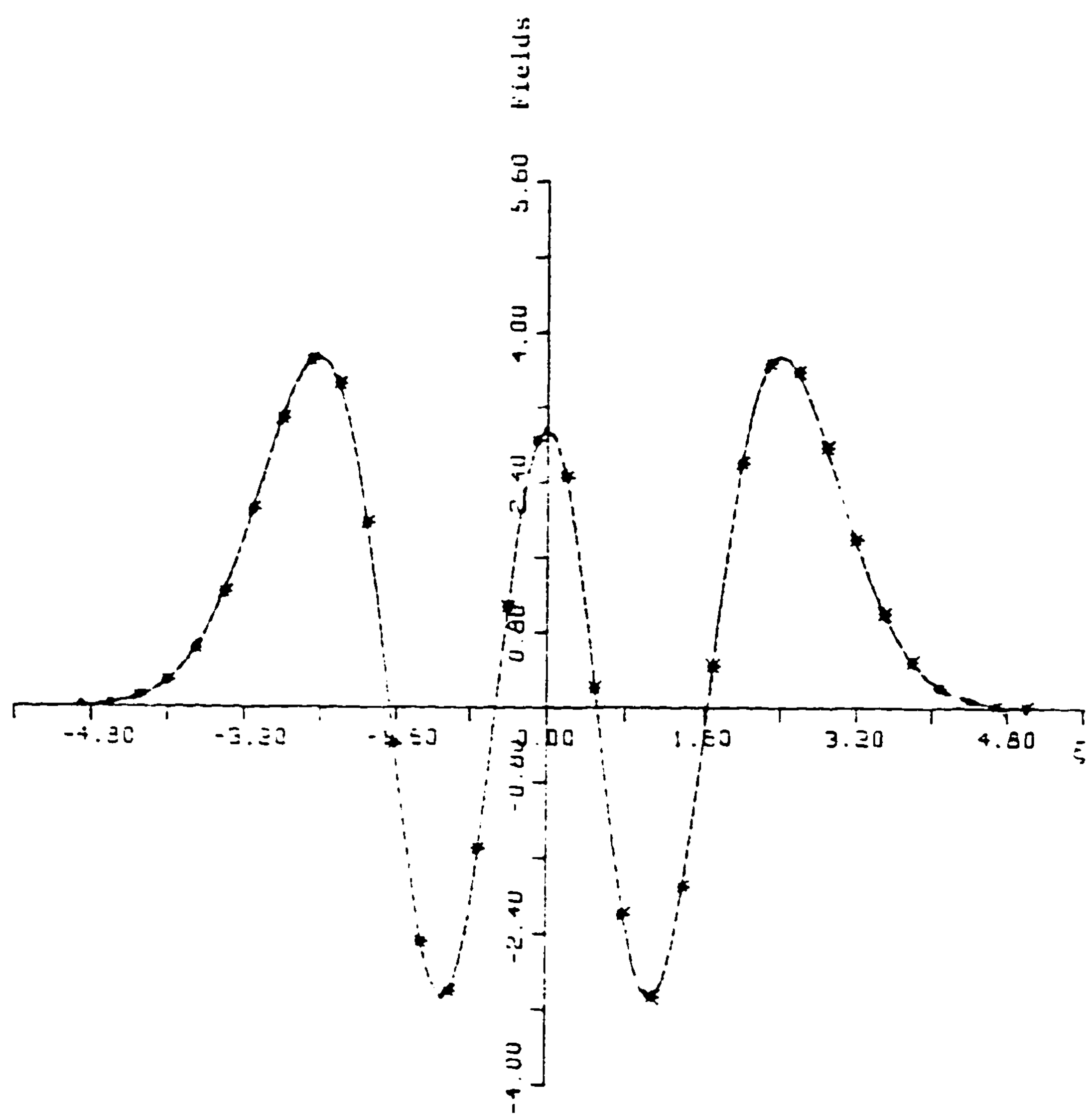
(c)

Second mode (n=2) for p=4.



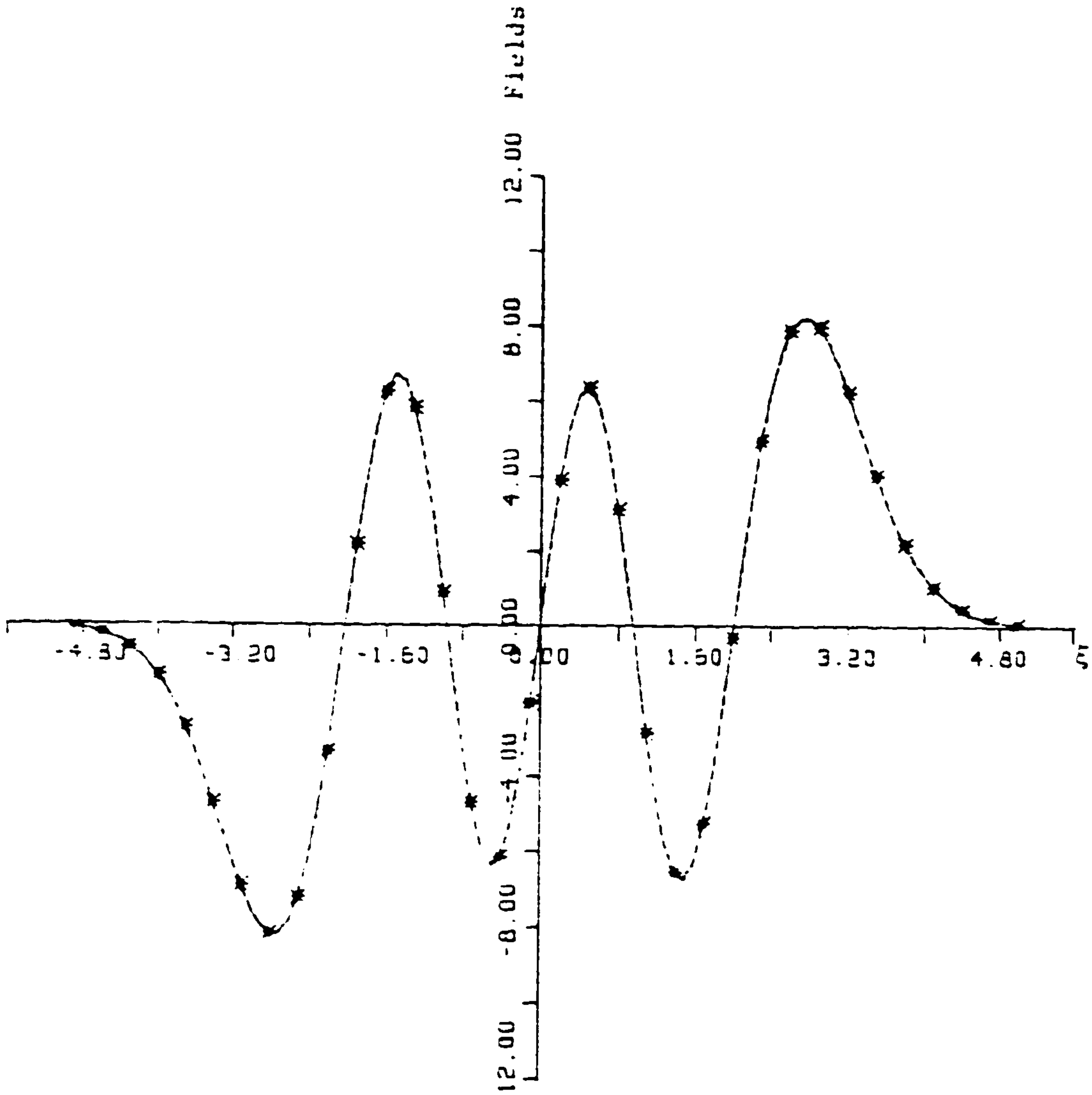
(d)

Third mode (n=3) for p=4.



(e)

Fourth mode (n=4) for p=4.



(f)

Fifth mode (n=5) for p=4.

TABLE IV. Comparison of Eigenvalues.

$\delta_1 = 0.0015$ $\delta_3 = 0.0025$

$\delta_2 = 0.0030$ $\delta_4 = 0.0045$

Mode Count	Unperturbed Eigenvalues	Perturbed Eigenvalues 'b_r' Linear Perturbation p=1	Cubic Perturbation p=3	Quartic Perturbation p=4
0	0.0002000	0.0001149	0.0001974	0.0002148
1	0.0006000	0.0005916	0.0005923	0.0006744
2	0.0010000	0.0009996	0.0009872	0.0011942
3	0.0014000	0.0014003	0.0013824	0.0017741
4	0.0018000	0.0018084	0.0017828	0.0024160
5	0.0022000	0.0022851	0.0022578	0.0031165

are plotted, for the case of linear perturbation ($p=1$), for $r=0$ through 5 respectively. The solid curve represents the perturbed field 'F_r', and the dotted curve is a plot of 'F_n' for comparison in each case. Figures 3 and 4 show plots for $p=3$ and $p=4$ respectively.

From an examination of eigenvectors for each case, it can be seen that for a particular r, the coefficient A_{rn} with $r=n$ is close to unity, whereas when $r \neq n$, $|A_{rn}| << 1$. eg., in Table I for linear ($p=1$) perturbation.

$A_{00} = 0.9117417$ while $A_{02} = 0.0906909$ and so on. This shows the validity of the assumption that the perturbed fields are close in magnitude to the unperturbed fields, and that the series in Eqn. (3-3-2) can be truncated at a suitable value of n without causing any significant error. Thus in the case of the fundamental perturbed mode $r=0$, (for $p=1$), the magnitude of the fifth coefficient is $\frac{0.0016}{0.92} \simeq 0.174\%$ of the first term i.e., a contribution of less than two parts in a thousand. This fact can also be seen from each of the plots, where the solid curve (perturbed field) mostly lies close to the dotted curve (unperturbed field), and the list of eigenvalues in Table IV. Hence to describe a particular perturbed mode, only a small number of unperturbed modes is required.

In each of the three cases, the last modes namely $r=4$ and $r=5$ are relatively less accurate than the first modes. The reason for this inaccuracy is that the matrix A (which is actually infinite) has been truncated at a value $N=5$. Although the contribution of the terms which have been neglected is negligible for the case when r=0 to 3, they do have some significance for r=4 and r=5. To check the validity of this fact, the program was also run for N=4 and N=3. The results showed a little change (in the 6th decimal place) for the coefficients A_{rn} in the last row and column of the eigenvector matrix, but in all the cases (i.e., p=1, 3 and 4), with N=4 and N=3, the rest of the coefficients A_{rn} were unaltered (as in Table III(b) and (c)). Thus increased accuracy for these modes (r=4 and 5) can be obtained by increasing N.

In Table III for the quartic (p=4) perturbation, it can be noted that half of the coefficients A_{rn} are zero. This, in general, will be the case for all even values of p, the reason being that for the case of an even perturbed mode (r=0,2,4, etc.), the contribution of the unperturbed modes corresponding to odd values of n is zero. A similar reason can be given for the coefficients of odd perturbed modes (r=1,3,5, etc). This is analogous to the fact that the Fourier series representation of an even periodic function will contain no sine terms (which represent odd functions) and vice versa. The above mentioned phenomenon does not take place for the case of odd perturbations (p=1 and 3) because then the refractive index profile is neither completely even nor completely odd.

Finally we tabulate (Table V) the eigenvalues from case (c) (namely quartic perturbation or p=4) of our examples along with those reported by Hashimoto |42| for the sake of comparison. Also shown

TABLE V. Comparison of Results for p=4.

$\delta_2 = 0.003$; $\delta_4 = 0.0045$; $\frac{g}{k} = 2 \times 10^{-4}$

Mode Count 'r'	Perturbed Eigenvalues 'b_r' for p=4		
	Hashimoto's Results	Equation (3-3-14)	Exact WKB
0	2.1238×10^{-4}	2.1480×10^{-4}	2.0703×10^{-4}
1	6.5438×10^{-4}	6.7440×10^{-4}	6.5708×10^{-4}
2	11.1812×10^{-4}	11.9420×10^{-4}	11.4612×10^{-4}
3	15.7812×10^{-4}	17.7410×10^{-4}	16.6744×10^{-4}
4	-	24.1600×10^{-4}	-
5	-	31.1650×10^{-4}	-

are the eigenvalues calculated by using the WKB method (and reported by |42| as well). From this table, it can be noted that when r=0, the results from Eqn. (3-3-14) are close to Hashimoto's results, and the difference is relatively large in comparison with the WKB results. However for higher valued modes, the results from Eqn. (3-3-14) are close to the WKB results (as compared to Hashimoto's results). The reason for this is that the WKB method gives relatively less dependable results near the turning point, and the mode which is most confined to the axis is the zeroth mode. It must be noted that |42| has only treated the perturbation corresponding to p=4 and has reported only the first four modes.

4.0 ELECTROMAGNETIC WAVE PROPAGATION IN A CLADDED INHOMOGENEOUS MEDIUM

In this chapter, a power series method is developed by combining the work of Dil and Blok |40| and Vassel |41| to investigate the propagation of electromagnetic waves along a radially inhomogeneous optical fiber. The method handles all the four tangential components of the fields simultaneously. It has been applied to cases where we have perturbations of a polynomial type on the parabolic refractive index profile. The reason for the cylindrical coordinates used in this chapter is that almost all optical fibers being produced presently have a cylindrical geometry. Solutions within the core are set up in terms of a power series, and the coefficients of the power series are calculated by a computer program for the case of propagating TE and TM modes. The solution in the homogeneous cladding is expressed in terms of modified Bessel functions. The process of matching the core solution to the cladding solution at the interface results in a characteristic equation, the solutions of which are the unknown propagation constants.

4.1 BASIC THEORY

The term "optical fiber" describes a certain type of dielectric waveguide for guiding light waves. These waveguides are called fibers because of their filamentary appearance. The type of dielectric waveguide that is depicted in Fig. 5 is called a cladded

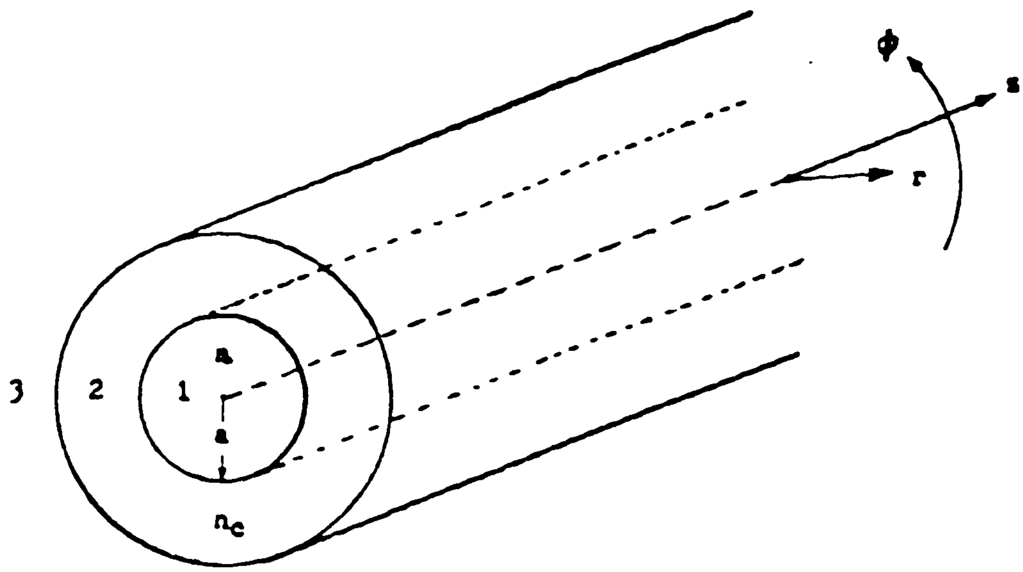


Figure 5. An optical-fiber.

fiber. A dielectric cylinder of refractive index n is surrounded by a concentric dielectric cylinder of refractive index n_c, where $n > n_c$ always. For the case when n does not vary with radius, we have a step-index fiber, but when n is a function of radius, we are dealing with a graded-core fiber. The second region, with index n_c, is not really essential for the principle of wave guidance in this type of dielectric waveguide. The optical fiber functions even when $n_c = 1$ (i.e., region 2 is air).

There are two reasons why it is desirable to use a cladded optical fiber rather than a bare dielectric cylinder. The fields of a dielectric waveguide are not fully contained inside the dielectric region with index n, but they extend into the outside region where they decay exponentially. Since the fiber must somehow be supported in space, it is advantageous to surround the inner core with an outer cladding to avoid scattering and field distortion by the supporting mechanism coming in touch with the guided field. However as the field decays exponentially inside region 2, practically no fields exist outside the cladding and we usually assume the cladding to extend to infinity for the purpose of calculations. The second reason for using cladded fibers has to do with the mechanism of field guidance in the fiber. At a given frequency, an optical fiber is capable of supporting a finite number of modes |50|. If the diameter of the core is much larger than the wavelength of the guided radiation, a very large number of guided modes is possible. It is often desirable to limit the number of guided modes and keep it as small as possible. Single guided mode operation is possible by properly dimensioning the guide. The dimensions of the inner core that permit single mode operation depend critically on the ratio n/n_c.

4.2 ANALYSIS OF A RADIALLY INHOMOGENEOUS OPTICAL-FIBER

The method of solution of the electromagnetic equations for guides of circular cross-section is similar to that followed for rectangular guides. However in order to simplify the application of boundary conditions it is convenient to express the field equations in the cylindrical coordinate system (i.e., r, ϕ, z).

Assume that the field distributions in a cylindrical dielectric waveguide (like the one shown in Fig. 5) are of the form

$$\begin{bmatrix} \overline{E}\ (r,\phi,z,t) \\ \\ \overline{H}\ (r,\phi,z,t) \end{bmatrix} = \begin{bmatrix} \overline{E}(r) \\ \\ \overline{H}(r) \end{bmatrix} \cdot \exp\left[j(\omega t - m\phi - \beta z)\right] \qquad (4\text{-}2\text{-}1)$$

where β is the propagation constant in the z direction and ω is the angular frequency. Furthermore because $\overline{E}$ and $\overline{H}$ must be single valued functions of position, they must be periodic in ϕ with period 2π. The polar index m which can thus have only integral values, expresses this periodicity.

On substituting eqn. (4-2-1) in Maxwell's Equations (2-2-1) and (2-2-2), the components E_r, E_ϕ, E_z, H_r, H_ϕ and H_z of the fields $\overline{E}(r)$ and $\overline{H}(r)$ are seen to satisfy the following relations for a medium of zero conductivity:

$$\left.\begin{aligned}
\frac{m}{r} H_z - \beta H_\phi &= -\omega\, \varepsilon_o\, n^2 E_r \\
\frac{m}{r} E_z - \beta E_\phi &= \omega\, \mu_o\, H_r \\
j\beta H_r + \frac{d}{dr} H_z &= -j\omega\varepsilon_o n^2\, E_\phi \\
j\beta E_r + \frac{d}{dr} E_z &= j\omega\mu_o\, H_\phi \\
\frac{1}{r}\,\frac{d}{dr}(r\, H_\phi) + j\,\frac{m}{r} H_r &= j\omega\varepsilon_o\, n^2\, E_z \\
\frac{1}{r}\,\frac{d}{dr}(r\, E_\phi) + j\,\frac{m}{r} E_r &= -j\omega\mu_o\, H_z
\end{aligned}\right\} \qquad (4\text{-}2\text{-}2)$$

where we have used $\varepsilon = \varepsilon_o n^2$, and the refractive index n is a general function of r. For the present case, n will represent the perturbed square-law medium of eqn. (3-3-1) with x replaced by r within the core so that (see Fig. 6).

$$n^2(r) = \left\{\begin{array}{lcl} n_o^2\left[1-\delta_2\left(\frac{r}{a}\right)^2 - \delta_p\left(\frac{r}{a}\right)^p\right] & ; & r<a \\ \\ n_c^2 & ; & r>a \end{array}\right\} \qquad (4\text{-}2\text{-}3)$$

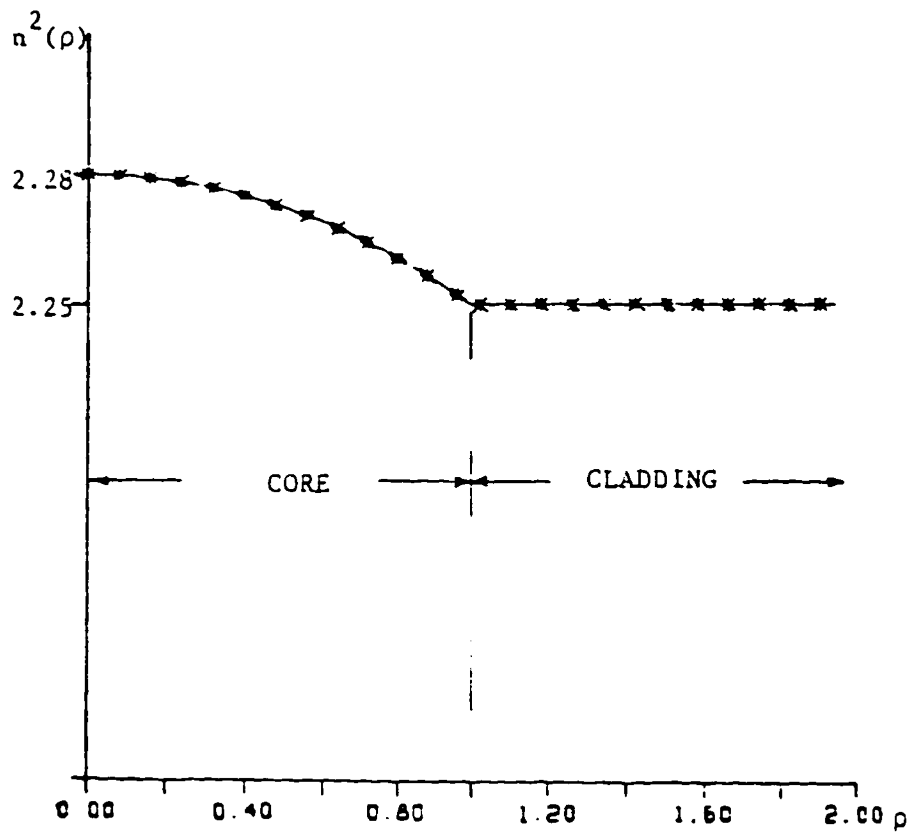


Figure 6. Parabolic refractive-index profile for an optical-fiber ($\delta_2 = 0.003$ and $\delta_p = 0.0$).

The first and second equations of the above set may be used to eliminate E_r and H_r in favour of the tangential field components. This is necessary because a discontinuity in the material properties occurs at $r = a$ and hence it is advantageous to deal with field quantities that are continuous across this boundary —— so that the application of boundary conditions is made easier.

In order to simplify the analysis, the following substitutions are made;

$\alpha = \beta/k_o$; where $k_o = \frac{2\pi}{\lambda_o} = \omega\sqrt{\mu_o\varepsilon_o}$ is the free space wavenumber and λ_o is the free-space wavelength. Furthermore it proves useful to introduce the dimensionless variable $\rho = r/a$ so that $n(r) = n(\rho)$, and $\frac{d}{dr} = \frac{1}{a}\frac{d}{d\rho}$ together with redefinitions of functions:

$$\left.\begin{aligned}
F_1(\rho) &= E_z(r) \\
F_2(\rho) &= -j\rho\cdot\sqrt{\frac{\mu_o}{\varepsilon_o}}\cdot H_\phi(r) \\
F_3(\rho) &= \rho\, E_\phi(r) \\
F_4(\rho) &= -j\cdot\sqrt{\frac{\mu_o}{\varepsilon_o}}\cdot H_z(r)
\end{aligned}\right\} \qquad (4\text{-}2\text{-}4)$$

Thus eqns. (4-2-2) transform to a new matrix equation of the form

$$\frac{d}{d\rho} \cdot F(\rho) = \frac{1}{\rho} \cdot A(\rho) \cdot F(\rho) \tag{4-2-5}$$

F is a column vector whose four elements determine the tangential components of the fields, and the 4 x 4 matrix $A(\rho)$ has analytic elements given by

$$A(\rho) = \begin{bmatrix} 0 & c(\frac{\alpha^2}{n^2} - 1) & 0 & \frac{-m\alpha}{n^2} \\ (c\rho^2 n^2 - \frac{m^2}{c}) & 0 & m\alpha & 0 \\ 0 & \frac{m\alpha}{n^2} & 0 & (c\rho^2 - \frac{m^2}{cn^2}) \\ -m\alpha & 0 & c(\alpha^2 - n^2) & 0 \end{bmatrix} \tag{4-2-6}$$

with $c = ak_o$.

From an examination of eqn. (4-2-6), it is obvious that the system of eqn. (4-2-5) has a regular singularity at $\rho = 0$. It is known [51,52] that every eigensolution of such a system has the form

$$F(\rho) = \rho^{\lambda} \cdot L(\rho) \tag{4-2-7}$$

where the 4-component vector $L(\rho)$ is regular, and can be expressed in the form of a power series in ρ. Also $L(o) \neq o$. The exponent λ

is determined by the constant term $A^o \equiv A(o)$ in the power series expansi.. of $A(\rho)$ in terms of ρ.

Substituting (4-2-7) in (4-2-5) yields

$$\rho \frac{dL(\rho)}{d\rho} = \left[A(\rho) - \lambda I\right] \cdot L(\rho) \tag{4-2-8}$$

where I is a 4 x 4 identity matrix. At $\rho = 0$, (4-2-8) reduces to

$$A^o \cdot L^o = \lambda L^o \tag{4-2-9}$$

Thus λ is the eigenvalue of the constant matrix A^o, and the corr..sponding eigenvector is $L^o \equiv L(o)$, where A^o is given by (n_o is the value of the refractive index at $\rho = o$)

$$A^o = A(o) = \begin{bmatrix} 0 & c\left(\frac{\alpha^2}{n_o^2} - 1\right) & 0 & -\frac{m\alpha}{n_o^2} \\ -\frac{m^2}{c} & 0 & m\alpha & 0 \\ 0 & \frac{m\alpha}{n_o^2} & 0 & -\frac{m^2}{cn_o^2} \\ -m\alpha & 0 & c(\alpha^2 - n_o^2) & 0 \end{bmatrix} \tag{4-2-10}$$

Applying the determinantal condition

$$|A^o - \lambda I| = 0$$

one can find four eigenvalues of the matrix A^o. They are $\lambda = m,m,-m,-m$. Of these four, only the degenerate pair $\lambda = m,m$ corresponds to physically admissible solutions i.e., proper or bounded solutions. This is not surprising because bounded solutions for F_1, for example, are expected to have behaviour near $\rho = 0$ similar to that of F_1 (or E_z) for a homogeneous cylinder of refractive index n_o. The solution of the homogeneous index case is known to be the Bessel function J_m |53| which varies as ρ^m for small values of ρ.

The two-fold degeneracy of the proper eigenvalues $\lambda = m,m$ implies that there are two arbitrary constants associated with the core solution. This means, for example, that the first and fourth components of L^o may be set equal to $(n_o^2 - \alpha^2)\, P_1$ and $(n_o^2 - \alpha^2)\, P_2$ respectively, where P_1 and P_2 are arbitrary constants. The first and fourth component equations from (4-2-9) with $\lambda = m$, may then be used to express the second and third components of L^o in terms of P_1 and P_2.

The two linearly independent eigenvectors of eqn. (4-2-9) become

$$L_A^o = L_A(o) = \begin{bmatrix} \alpha^2 - n_o^2 \\ \dfrac{mn_o^2}{2} \\ \dfrac{m\alpha}{c} \\ 0 \end{bmatrix} \text{ and } L_B^o = L_B(o) = \begin{bmatrix} 0 \\ \dfrac{m\alpha}{c} \\ \dfrac{m}{c} \\ \alpha^2 - n_o^2 \end{bmatrix} \qquad (4\text{-}2\text{-}11)$$

By this means, the total solution $L(\rho)$ at the origin (ρ=0) is expressible as a linear combination

$$L(0) \equiv L^{o} = P_1 L_A^{o} + P_2\, L_B^{o} \qquad (4\text{-}2\text{-}12)$$

Now since the differential eqn. (4-2-8) and the initial conditions are linear, $L(\rho)$ and hence $F(\rho)$ can be decomposed with the same constants P_1 and P_2. Thus

$$F(\rho) = P_1 F_A(\rho) + P_2 F_B(\rho); \qquad 0 \leq \rho < 1 \qquad (4\text{-}2\text{-}13)$$

where

$$F_A(\rho) = \rho^m \cdot L_A(\rho)$$

and

$$F_B(\rho) = \rho^m \cdot L_B(\rho) \qquad (4\text{-}2\text{-}14)$$

4.3 POWER SERIES EXPANSION OF $L(\rho)$

The statement of the problem thus developed is now open to numerical analysis. The fact that a variety of approximate treatments have been reported indicates that rigourous analysis of the natural modes of a radially inhomogeneous medium is difficult to carry out. Basically one solves one fourth-order differential equation or two coupled second-order ones. The scheme of the previous section, however, aims at solving four coupled first order differential equations as a system of simultaneous equations. In all cases no simplifying assumptions are made.

The majority of approaches do in fact make some simplifying assumptions so that the two second order differential equations are uncoupled. The most popular of the assumptions is that the logarithm of the permittivity of the medium changes only very slowly over distances of the order of a wave length. As a result the electric and magnetic field vectors separate, and one would obtain solutions by one of the approximate methods, e.g. the variational method, the WKB method or the perturbation method. The latter of these methods has been demonstrated in this work in chapter 3.

In the present chapter Maxwell's equations are kept intact and a power series expansion is assumed for the solution with the aim of testing its applicability, and use it, where applicable, to calculate the effect of perturbation. The basic assumption is that the vector $L(\rho)$ is expressible in the form of a power series (in powers of ρ) within the interval $0 \leq \rho < 1$ i.e.,

$$L(\rho) = \sum_{i=0}^{\infty} L^i \cdot \rho^i \tag{4-3-1}$$

where L^i are constant 4×1 vectors (i.e., coefficients).

Moreover $A(\rho)$ can also be written as

$$A(\rho) = \sum_{i=0}^{\infty} A^i \rho^i \tag{4-3-2}$$

where A^i are constant 4 x 4 matrices which can be calculated. Then by inserting (4-3-1) and (4-3-2) into (4-2-8) (with $\lambda = m$), we get

$$\rho \cdot \left[\sum_{i=0}^{\infty} i \cdot L^i \cdot \rho^{i-1} \right] = \left[\sum_{i=0}^{\infty} A^i \rho^i - mI \right] \left[\sum_{i=0}^{\infty} L^i \rho^i \right] \tag{4-3-3}$$

Equating the coefficients of each power of ρ, gives

$$\left.\begin{aligned} &\left[A^o - mI\right] \cdot L^o = 0 && ;\ \text{for } i=0 \\ &\left[A^o - (m+i)I\right] \cdot L^i = - \sum_{j=1}^{i} A^j \cdot L^{i-j} && ;\ \text{for } i \geq 1 \end{aligned}\right\} \tag{4-3-4}$$

The first equation is just a repetition of eqn. (4-2-9), whereas the second equation can be inverted by taking advantage of the fact that $(m+i)$ is not an eigenvalue of A^o for $i \geq 1$. Thus $\left[A^o - (m+i)\, I\right]$ is non-singular and its inverse exists. Moreover it can be shown that

$$\left[A^o\right]^2 = m^2 I.$$

From this it follows that

$$\left[A^o - (m+i)I\right] \cdot \left[A^o + (m+i)I\right] = -i(i+2m)I \tag{4-3-5}$$

This implies that

$$\left[A^{o} - (m+i)I\right]^{-1} = \frac{-1}{i(i+2m)} \cdot \left[A^{o} + (m+i)I\right] \qquad (4\text{-}3\text{-}6)$$

Hence the second equation in (4-3-4) gives

$$L^{i} = \frac{1}{i(i+2m)} \cdot \left[A^{o} + (m+i)I\right] \cdot \sum_{j=1}^{i} A^{j} \cdot L^{i-j} \qquad (4\text{-}3\text{-}7)$$

Equation (4-3-7) allows us to calculate the successive vector coefficients from L^{0} by simple matrix multiplication. Thus by setting L^{o} equal to L_A^{o} and L_B^{o} , in turn, we can generate formal power series expansions for the linearly independent vectors $L_A(\rho)$ and $L_B(\rho)$.

4.4 FIELDS IN THE CLADDING

Outside the core, i.e., for $\rho>1$, the medium is homogeneous and

$$n(\rho) = n_c \qquad (4\text{-}4\text{-}1)$$

From eqn. (4-2-5), the following equation is obtained;

$$\frac{1}{\rho} \cdot \frac{d}{d\rho}\left[\rho \cdot \frac{d}{d\rho} \cdot F(\rho)\right] = \frac{1}{\rho} \cdot \frac{d}{d\rho} \cdot \left[A(\rho) \cdot F(\rho)\right] \qquad (4\text{-}4\text{-}2)$$

The first equation in this set is

$$\frac{1}{\rho} \cdot \left[\frac{dF_1}{d\rho} + \rho \frac{d^2F_1}{d\rho^2}\right] = \frac{1}{\rho} \cdot \left[c\left(\frac{\alpha^2}{n_c^2} - 1\right) \cdot \frac{dF_2}{d\rho} - \frac{m\alpha}{n_c^2} \cdot \frac{dF_4}{d\rho}\right] \qquad (4\text{-}4\text{-}3)$$

In order to obtain an equation in F_1 only, use is made of eqn. (4-2-5) whereby the following two equations are obtained;

$$\frac{dF_2}{d\rho} = \frac{1}{\rho} \cdot \left[\left(c\rho^2 n_c^2 - \frac{m}{c}\right) F_1 + m\alpha F_3\right],$$

and

$$\frac{dF_4}{d\rho} = \frac{1}{\rho} \cdot \left[-m\alpha F_1 + c(\alpha^2 - n_c^2) F_3\right].$$

Using these in (4-4-3) and after some simplification,

$$\frac{d^2 F_1}{d\rho^2} + \frac{1}{\rho} \frac{dF_1}{d\rho} - \left(b_1^2 + \frac{m^2}{\rho^2}\right) F_1 = 0 \qquad (4\text{-}4\text{-}4)$$

where $b_1^2 = c^2 (\alpha^2 - n_c^2)$

Equation (4-4-4) can easily be recognized as the Modified Bessel's Equation |54|. To ensure that the energy is confined to the vicinity of the core, and additional boundary condition that the fields must decay as $\rho \to \infty$ is imposed. Modes are referred to as "proper" when this condition is satisfied. The waves in the homogeneous cladding are evanescent in nature, and the corresponding modes in the core that tend towards the core-cladding surface are the well-known "surface waves". If the condition that the fields must vanish as $\rho \to \infty$, is not made the corresponding fields increase exponentially with ρ for $\rho>1$. These are known as "improper modes". Thus a physically admissible solution of (4-4-4) is

$$F_1 = -P_3 \cdot K_m (b_1\rho) \tag{4-4-5}$$

where P_3 is an arbitrary constant and K_m is the Modified Bessel function of the second kind, of order m, with real argument. A similar process can be repeated for F_4, to give

$$F_4 = -P_4 \cdot K_m (b_1\rho) \tag{4-4-6}$$

Note that because b_1 is to be real, $|\alpha|$ must always be greater than n_c.

Because of the two linearly independent solutions in the core, one also looks for two solutions F_C and F_D in the cladding. By using (4-2-5), (4-4-5) and (4-4-6) and a similar argument as used for the core solutions L_A and L_B, it is found that

$$F_C(\rho) = \begin{bmatrix} K_m\ (b_1\rho) \\ \\ \dfrac{cn_c^2\rho}{b_1} \cdot K_m'\ (b_1\rho) \\ \\ \dfrac{cm\alpha}{b_1^2} \cdot K_m\ (b_1\rho) \\ \\ 0 \end{bmatrix} \qquad (4\text{-}4\text{-}7)$$

when the following conditions are used; $F_1 = -P_3 \cdot K_m(b_1\rho)$ and $F_4 = 0$. If on the other hand the conditions $F_1 = 0$ and $F_4 = -P_4\ K_m(b_1\rho)$ are used one obtains

$$F_D(\rho) = \begin{bmatrix} 0 \\ \\ \dfrac{cm\alpha}{b_1^2} \cdot K_m(b_1\rho) \\ \\ \dfrac{c\rho}{b_1} \cdot K_m'(b_1\rho) \\ \\ K_m(b_1\rho) \end{bmatrix} \qquad (4\text{-}4\text{-}8)$$

The primes in the above equations denote differentiation with respect to the argument $b_1\rho$ of the Bessel function. Finally the general solution in the cladding is given by

$$F(\rho) = -P_3F_C(\rho) - P_4F_D(\rho) \ ; \ \rho > 1 \qquad (4\text{-}4\text{-}9)$$

4.5 BOUNDARY CONDITIONS AT $\rho = 1$

Since the components of the vector $F(\rho)$ are continuous across the boundary, at $\rho = 1$, the solution of eqn. (4-2-5) in the region $0 \le \rho < 1$ must be equal to its solution in the region $\rho > 1$. This statement is a result of the continuity condition of the tangential field components across the boundary of two different media, implying that

$$P_1 F_A(\rho) + P_2 F_B(\rho) + P_3 F_C(\rho) + P_4 F_D(\rho) = 0 \qquad (4\text{-}5\text{-}1)$$

Equation (4-5-1) represents a system of four equations in the four unknowns P_1, P_2, P_3 and P_4 in addition to a parameter α. The parameter α which is the normalized propagation constant can be determined from the fact that the determinant of this system should be zero if a non-trivial solution exists i.e.,

$$\begin{vmatrix} L_{A1} & L_{B1} & K_m & 0 \\ L_{A2} & L_{B2} & \frac{cn_c^2}{b_1} K_m' & \frac{cm\alpha}{b_1^2} \cdot K_m \\ L_{A3} & L_{B3} & \frac{cm\alpha}{b_1^2} K_m & \frac{c}{b_1} \cdot K_m' \\ L_{A4} & L_{B4} & 0 & K_m \end{vmatrix} = 0 \qquad (4\text{-}5\text{-}2)$$

where L_{A1} represents the first element of vector L_A and so on. Moreover all the elements in the above determinant have been evaluated at $\rho = 1$. Expanding this determinant gives the following characteristic equation in which $\gamma_m = b_1 \dfrac{K'_m(b_1)}{K_m(b_1)}$ and the primes denote differentiation of the Bessel function with respect to its argument b_1,

$$\left\{\left(\frac{b_1^2}{c} \cdot L_{A2} - m\alpha L_{A4} - n_c^2 \cdot \gamma_m \cdot (L_{A1})\right) \cdot \left(\frac{b_1^2}{c} \cdot L_{B3} - m\alpha L_{B1} - \gamma_m \cdot L_{B4}\right) - \left(\frac{b_1^2}{c} \cdot L_{B2} - m\alpha L_{B4} - n_c^2 \cdot \gamma_m \cdot L_{B1}\right) \cdot \left(\frac{b_1^2}{c} \cdot L_{A3} - m\alpha L_{A1} - \gamma_m \cdot L_{A4}\right)\right\} = 0 \tag{4-5-3}$$

In the above transcendental equation b_1, the L's and γ_m depend implicitly on α. Moreover the L's are only known numerically so that no general theory exists for solving this equation. The only possible way is to solve it numerically using some iterative interpolation technique.

4.6 SPECIAL CASE WHEN $m = 0$

The results developed in the previous sections show that in general when $m \neq 0$ the equations are coupled, and the solution of equation (4-2-5) represents "hybrid modes" even if the refractive index n is constant for $0 \leq \rho < 1$. In this case both the linearly independent solutions of this equation contain z components of the electric and magnetic fields |53|. If E_z makes a larger contribution, the mode is considered E-like and designated EH. Similarly if the contribution of H_z is greater, an HE mode or a H-like mode results. The use of two letters, such as EH and HE, to designate these modes is reasonable because it does imply their hybrid nature.

However, when m = 0, eqn. (4-2-5) reduces to two separate systems of two coupled first order differential equations which describe the TM and TE modes individually. The determinant of eqn. (4-5-2) can be rearranged to contain two 2 x 2 null determinants giving separate characteristic equations for TM and TE modes:

$$n_c^2 \, \gamma_m \, L_{A1} - \frac{b_1^2}{c} L_{A2} = 0 \; ; \text{ for TM modes} \qquad (4\text{-}6\text{-}1)$$

and

$$\gamma_m \cdot L_{B4} - \frac{b_1^2}{c} L_{B3} = 0 \; ; \text{ for TE modes} \qquad (4\text{-}6\text{-}2)$$

4.7 ILLUSTRATIVE EXAMPLES AND DISCUSSIONS

As in the previous chapter, the following examples are considered to illustrate the present theory.

(a) Unperturbed parabola

$$n^2(r) = \begin{cases} n_o^2\left[1 - \delta_2\left(\frac{r}{a}\right)^2\right] & ; r<a \\ n_c^2 & ; r>a \end{cases} \tag{4-7-1}$$

(b) Linear perturbation

$$n^2(r) = \begin{cases} n_o^2\left[1-\delta_2\left(\frac{r}{a}\right)^2 - \delta_1\left(\frac{r}{a}\right)\right] & ; r<a \\ n_c^2 & ; r>a \end{cases} \tag{4-7-2}$$

(c) Cubic perturbation

$$n^2(r) = \begin{cases} n_o^2\left[1-\delta_2\left(\frac{r}{a}\right)^2 - \delta_3\left(\frac{r}{a}\right)^3\right] & ; r<a \\ n_c^2 & ; r>a \end{cases} \tag{4-7-3}$$

(d) Quartic perturbation

$$n^2(r) = \begin{cases} n_o^2\left[1-\delta_2\left(\frac{r}{a}\right)^2 - \delta_4\left(\frac{r}{a}\right)^4\right] & ; r<a \\ n_c^2 & ; r>a \end{cases} \tag{4-7-4}$$

For the above four examples, the following constants were used

$n_o = 1.51$ $\qquad$ $n_c = 1.50$

$\delta_1 = 0.0003$ $\qquad$ $\delta_2 = 0.003$

$\delta_3 = 0.001$ $\qquad$ $\delta_4 = 0.002$

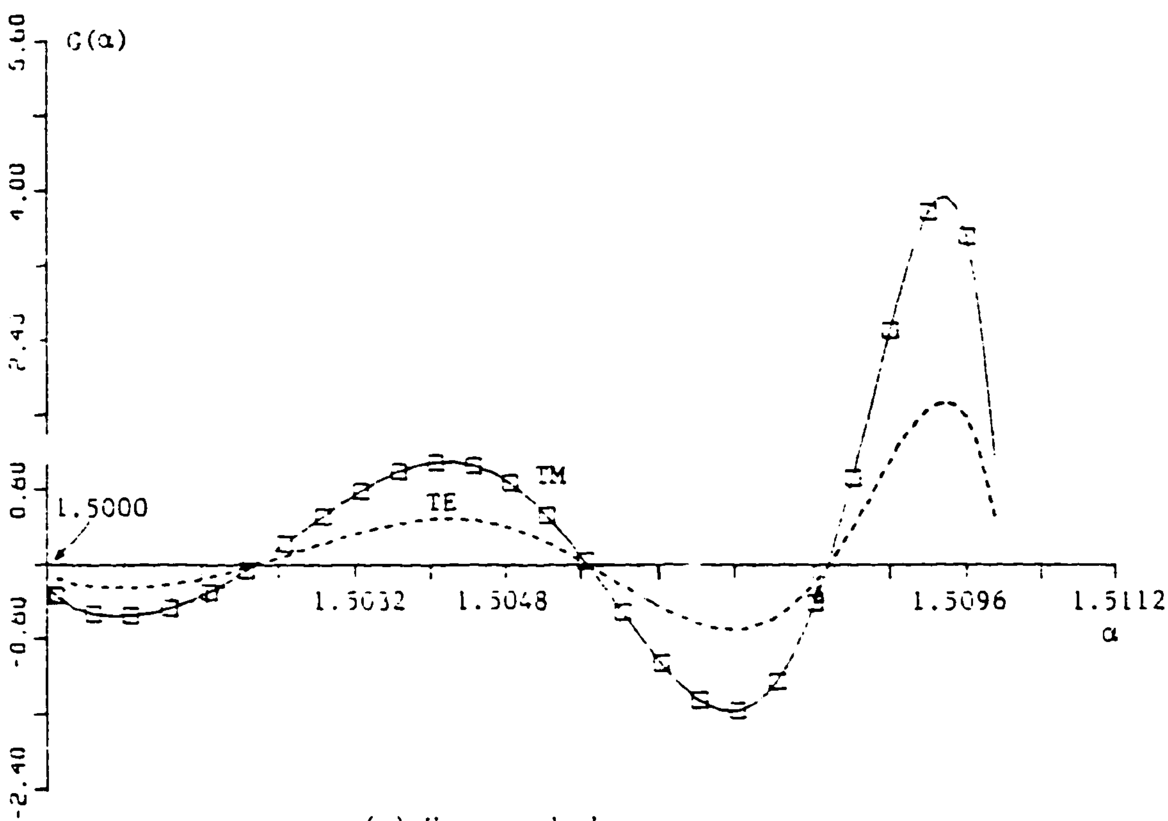
$\frac{a}{\lambda_o} = 10$ $\qquad$ $c = 20\pi$

Most of the numerical calculations were performed on the IBM 3033 using Fortran IV programs. All the computer programs that were written and used for this chapter are listed in Appendix A-2.

The first step is to calculate the values of α, by solving the implicit transcendental characteristic eqn. (4-5-3). To do this, the coefficients L_A's and L_B's in the power series expansion of the fields and the values of the modified Bessel functions K_m are needed. Note that when $m = 0$ (the only case for which we have calculated the fields and propagation constants), L_A describes TM modes and L_B describes TE modes, and are independent of each other as far as the calculations are concerned. However, both cases have been treated simultaneously in a single computer program. Evaluation of L_A and L_B was carried out using eqn. (4-3-1) and (4-3-7). Hence a knowledge of the coefficients A^i in the power series expansion of $A(\rho)$ is required. This information is supplied as input to the program for each of the examples mentioned above.

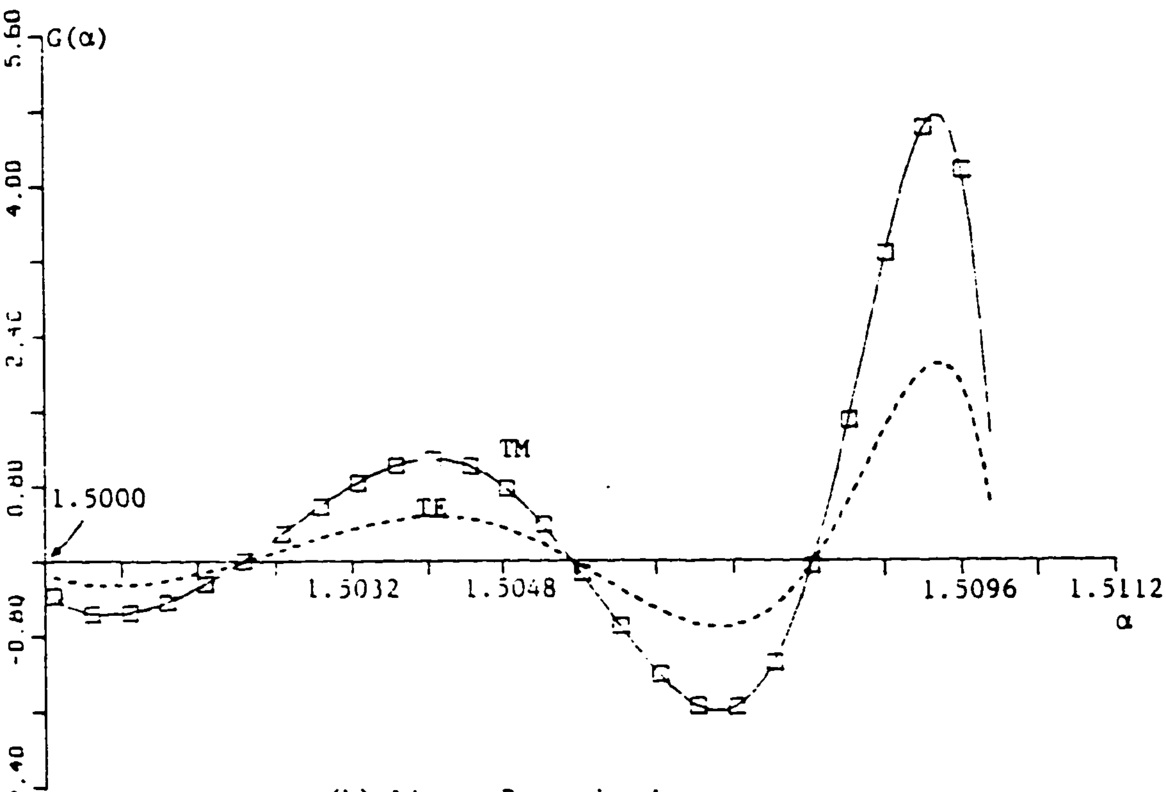
A difficult problem arises because of the fact that A^i, L_A, L_B, b_1, Y_m, K_m etc., all depend implicitly on α. Thus there is no general analytic way of solving the characteristic equation to find out the values of α. Naturally in such a situation one is forced to employ some graphical method. A rough graph of the left hand side (call it '$G(\alpha)$') is plotted versus α by taking 100 values of α, although this number can be varied. K_m and each of the coefficient matrices A^i, L_A, L_B, etc., are computed 100 times during the execution of the program, and after each computation, $G(\alpha)$ is evaluated, giving 100 points to plot. Finally, from the graph of $G(\alpha)$ versus α, the roots of the characteristic equation or the values of α when $G(\alpha) = 0$, for each case can be read.

Figure 7(a) through (d) shows the plots of $G(\alpha)$ versus α (for $1.50 < \alpha < 1.51$) for cases (a) through (d) respectively. The results of this figure are summarized in Table VI. Table VII(a) through (f) lists the values of the coefficients L^i in the power series expansions of the vectors L_A (TM) and L_B (TE) and plots of L_A and L_B can be seen in Fig. 8 (a) - (c) for the unperturbed case. Similarly results for linear, cubic and quartic perturbations (cases (b), (c) and (d)) are arranged in Table VIII, IX and X, and Figs. 9, 10 and 11 respectively. In all the plots mentioned above, solid curves represent TM modes and dotted curves represent TE modes, except for the case of Figs. 9, 10 and 11, where the solid curves represent the perturbed TE fields and the dotted curves represent

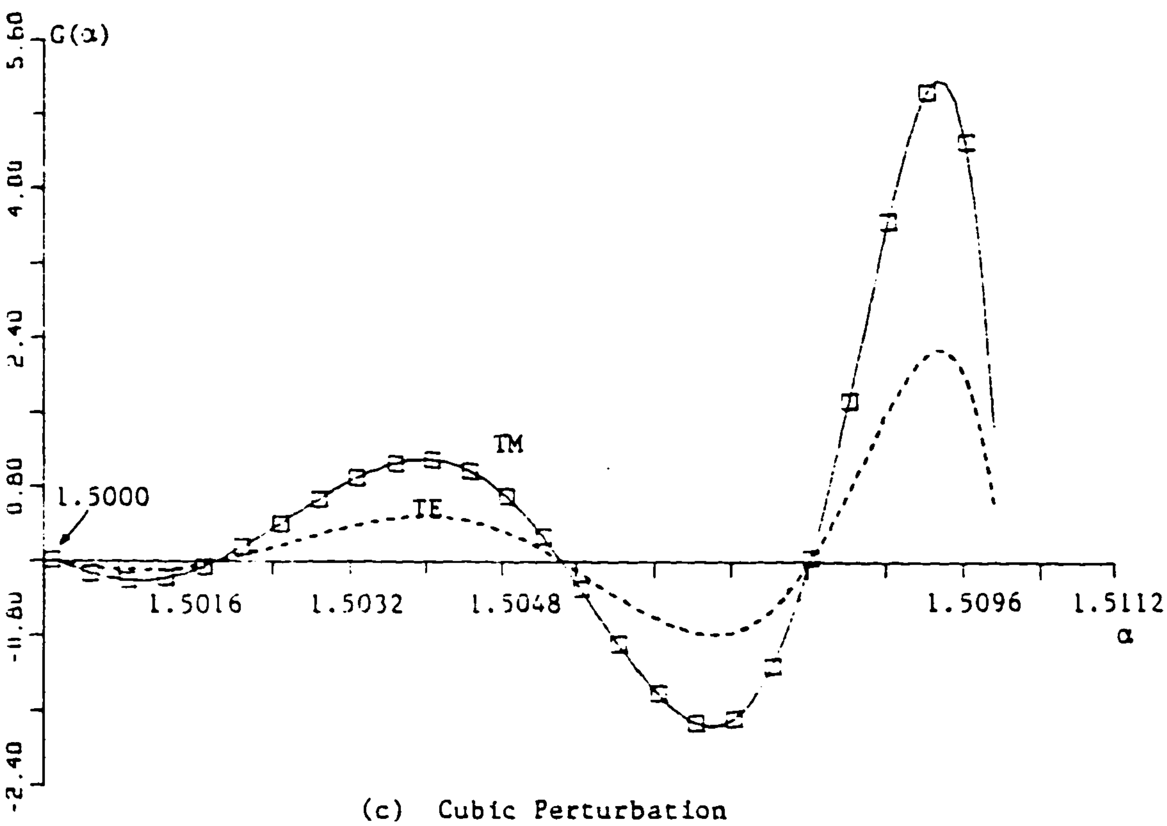


(a) Unperturbed case

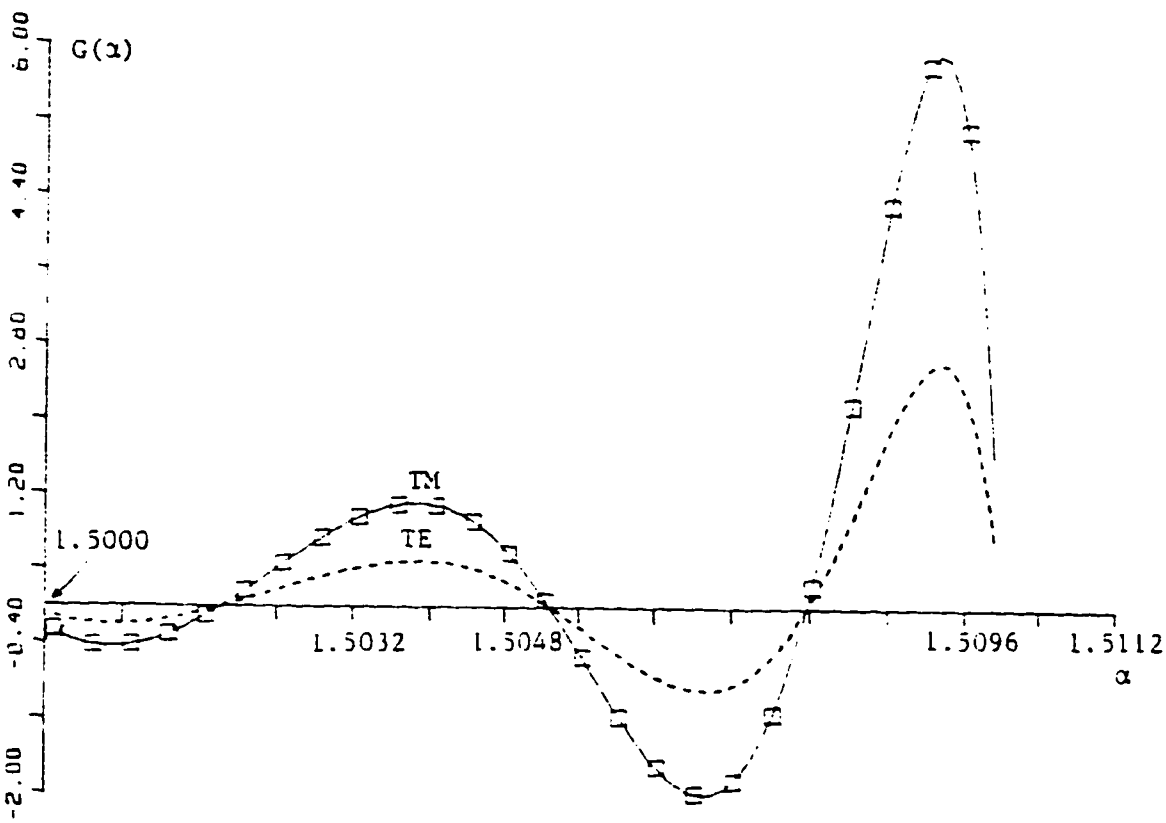
Figure 7. Plots of the Characteristic Equation versus α.



(b) Linear Perturbation.



(c) Cubic Perturbation



(d) Quartic Perturbation.

TABLE VI. Comparison of Eigenvalues for an Optical-Fiber

$n_o = 1.51$, $n_c = 1.50$, $c = 2\pi \times 10$,

$\delta_1 = 0.0003$ $\delta_2 = 0.003$ $\delta_3 = 0.001$ $\delta_4 = 0.002$

Mode Count	Unperturbed	Linear Perturbation	Cubic Perturbation	Quartic Perturbation
1	1.50816	1.50804	1.50802	1.50797
2	1.50569	1.50556	1.50546	1.50532
3	1.50216	1.50207	1.50180	1.50183

TABLE VII. Coefficients in the Series Expansion for the Unperturbed Case.

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50816

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	---->	-0.0055552	0.0	0.0	0.0
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0304555	-0.3979118	0.0	0.0
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	-0.0604699	1.0913496	0.0	0.0
L(5)	---->	0.0	0.0	0.0	0.0
L(6)	---->	0.0710923	-1.4459877	0.0	0.0
L(7)	---->	0.0	0.0	0.0	0.0
L(8)	---->	-0.0584081	1.2763214	0.0	0.0
L(9)	---->	0.0	0.0	0.0	0.0
L(10)	---->	0.0368542	-0.8398018	0.0	0.0
L(11)	---->	0.0	0.0	0.0	0.0
L(12)	---->	-0.0187982	0.4420646	0.0	0.0
L(13)	---->	0.0	0.0	0.0	0.0
L(14)	---->	0.0080529	-0.1934884	0.0	0.0
L(15)	---->	0.0	0.0	0.0	0.0
L(16)	---->	-0.0029685	0.0726076	0.0	0.0
L(17)	---->	0.0	0.0	0.0	0.0
L(18)	---->	0.0009610	-0.0238181	0.0	0.0
L(19)	---->	0.0	0.0	0.0	0.0
L(20)	---->	-0.0002771	0.0069475	0.0	0.0
L(21)	---->	0.0	0.0	0.0	0.0
L(22)	---->	0.0000721	-0.0018232	0.0	0.0
L(23)	---->	0.0	0.0	0.0	0.0
L(24)	---->	-0.0000171	0.0004351	0.0	0.0
L(25)	---->	0.0	0.0	0.0	0.0
L(26)	---->	0.0000037	-0.0000952	0.0	0.0
L(27)	---->	0.0	0.0	0.0	0.0
L(28)	---->	-0.0000007	0.0000192	0.0	0.0
L(29)	---->	0.0	0.0	0.0	0.0
L(30)	---->	0.0000001	-0.0000036	0.0	0.0

(a)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50816

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	---->	0.0	0.0	0.0	-0.0055552
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0	0.0	-0.1745151	0.0304555
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	0.0	0.0	0.4763795	-0.0604928
L(5)	---->	0.0	0.0	0.0	0.0
L(6)	---->	0.0	0.0	-0.6334600	0.0711154
L(7)	---->	0.0	0.0	0.0	0.0
L(8)	---->	0.0	0.0	0.5585222	-0.0583983
L(9)	---->	0.0	0.0	0.0	0.0
L(10)	---->	0.0	0.0	-0.3669165	0.0368104
L(11)	---->	0.0	0.0	0.0	0.0
L(12)	---->	0.0	0.0	0.1927330	-0.0187468
L(13)	---->	0.0	0.0	0.0	0.0
L(14)	---->	0.0	0.0	-0.0641327	0.0080141
L(15)	---->	0.0	0.0	0.0	0.0
L(16)	---->	0.0	0.0	0.0314702	-0.0029464
L(17)	---->	0.0	0.0	0.0	0.0
L(18)	---->	0.0	0.0	-0.0102845	0.0009508
L(19)	---->	0.0	0.0	0.0	0.0
L(20)	---->	0.0	0.0	0.0029870	-0.0002731
L(21)	---->	0.0	0.0	0.0	0.0
L(22)	---->	0.0	0.0	-0.0007800	0.0000707
L(23)	---->	0.0	0.0	0.0	0.0
L(24)	---->	0.0	0.0	0.0001852	-0.0000167
L(25)	---->	0.0	0.0	0.0	0.0
L(26)	---->	0.0	0.0	-0.0000403	0.0000036
L(27)	---->	0.0	0.0	0.0	0.0
L(28)	---->	0.0	0.0	0.0000081	-0.0000007
L(29)	---->	0.0	0.0	0.0	0.0
L(30)	---->	0.0	0.0	-0.0000015	0.0000001

(b)

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50569

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	---->	-0.0129995	0.0	0.0	0.0
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.1667743	-0.9311476	0.0	0.0
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	-0.5786507	5.9743614	0.0	0.0
L(5)	---->	0.0	0.0	0.0	0.0
L(6)	---->	1.0121775	-13.6280573	0.0	0.0
L(7)	---->	0.0	0.0	0.0	0.0
L(8)	---->	-1.1369324	18.1564789	0.0	0.0
L(9)	---->	0.0	0.0	0.0	0.0
L(10)	---->	0.9252782	-16.3310242	0.0	0.0
L(11)	---->	0.0	0.0	0.0	0.0
L(12)	---->	-0.5660120	11.0868883	0.0	0.0
L(13)	---->	0.0	0.0	0.0	0.0
L(14)	---->	0.3025752	-6.0249205	0.0	0.0
L(15)	---->	0.0	0.0	0.0	0.0
L(16)	---->	-0.1315783	2.7248964	0.0	0.0
L(17)	---->	0.0	0.0	0.0	0.0
L(18)	---->	0.0493554	-1.0544310	0.0	0.0
L(19)	---->	0.0	0.0	0.0	0.0
L(20)	---->	-0.0162634	0.3563569	0.0	0.0
L(21)	---->	0.0	0.0	0.0	0.0
L(22)	---->	0.0047758	-0.1068674	0.0	0.0
L(23)	---->	0.0	0.0	0.0	0.0
L(24)	---->	-0.0012644	0.0287986	0.0	0.0
L(25)	---->	0.0	0.0	0.0	0.0
L(26)	---->	0.0003047	-0.0070455	0.0	0.0
L(27)	---->	0.0	0.0	0.0	0.0
L(28)	---->	-0.0000673	0.0015781	0.0	0.0
L(29)	---->	0.0	0.0	0.0	0.0
L(30)	---->	0.0000138	-0.0003260	0.0	0.0

(c)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50569

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	---->	0.0	0.0	0.0	-0.0129995
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0	0.0	-0.4063803	0.1667744
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	0.0	0.0	2.6196079	-0.5787761
L(5)	---->	0.0	0.0	0.0	0.0
L(6)	---->	0.0	0.0	-6.0607462	1.0126705
L(7)	---->	0.0	0.0	0.0	0.0
L(8)	---->	0.0	0.0	7.9532585	-1.1375828
L(9)	---->	0.0	0.0	0.0	0.0
L(10)	---->	0.0	0.0	-7.1474276	0.9255853
L(11)	---->	0.0	0.0	0.0	0.0
L(12)	---->	0.0	0.0	4.8462067	-0.5858317
L(13)	---->	0.0	0.0	0.0	0.0
L(14)	---->	0.0	0.0	-2.6291256	0.3021533
L(15)	---->	0.0	0.0	0.0	0.0
L(16)	---->	0.0	0.0	1.1865177	-0.1311896
L(17)	---->	0.0	0.0	0.0	0.0
L(18)	---->	0.0	0.0	-0.4579245	0.0491084
L(19)	---->	0.0	0.0	0.0	0.0
L(20)	---->	0.0	0.0	0.1542739	-0.0161405
L(21)	---->	0.0	0.0	0.0	0.0
L(22)	---->	0.0	0.0	-0.0460957	0.0047251
L(23)	---->	0.0	0.0	0.0	0.0
L(24)	---->	0.0	0.0	0.0123699	-0.0012464
L(25)	---->	0.0	0.0	0.0	0.0
L(26)	---->	0.0	0.0	-0.0030120	0.0002991
L(27)	---->	0.0	0.0	0.0	0.0
L(28)	---->	0.0	0.0	0.0006711	-0.0000658
L(29)	---->	0.0	0.0	0.0	0.0
L(30)	---->	0.0	0.0	-0.0001378	0.0000134

(d)

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50216

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	---->	-0.0236158	0.0	0.0	0.0
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.5504021	-1.6915855	0.0	0.0
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	-3.2862844	19.7149963	0.0	0.0
L(5)	---->	0.0	0.0	0.0	0.0
L(6)	---->	9.1273975	-78.5041199	0.0	0.0
L(7)	---->	0.0	0.0	0.0	0.0
L(8)	---->	-15.1403294	163.623840	0.0	0.0
L(9)	---->	0.0	0.0	0.0	0.0
L(10)	---->	17.1924286	-217.290466	0.0	0.0
L(11)	---->	0.0	0.0	0.0	0.0
L(12)	---->	-14.5375595	205.789169	0.0	0.0
L(13)	---->	0.0	0.0	0.0	0.0
L(14)	---->	9.6811724	-149.287033	0.0	0.0
L(15)	---->	0.0	0.0	0.0	0.0
L(16)	---->	-5.2818928	87.0724487	0.0	0.0
L(17)	---->	0.0	0.0	0.0	0.0
L(18)	---->	2.4304867	-42.2687225	0.0	0.0
L(19)	---->	0.0	0.0	0.0	0.0
L(20)	---->	-0.9643891	17.5228882	0.0	0.0
L(21)	---->	0.0	0.0	0.0	0.0
L(22)	---->	0.3357370	-6.3273458	0.0	0.0
L(23)	---->	0.0	0.0	0.0	0.0
L(24)	---->	-0.1039861	2.0213194	0.0	0.0
L(25)	---->	0.0	0.0	0.0	0.0
L(26)	---->	0.0289814	-0.5785074	0.0	0.0
L(27)	---->	0.0	0.0	0.0	0.0
L(28)	---->	-0.0073373	0.1498761	0.0	0.0
L(29)	---->	0.0	0.0	0.0	0.0
L(30)	---->	0.0017010	-0.0354531	0.0	0.0

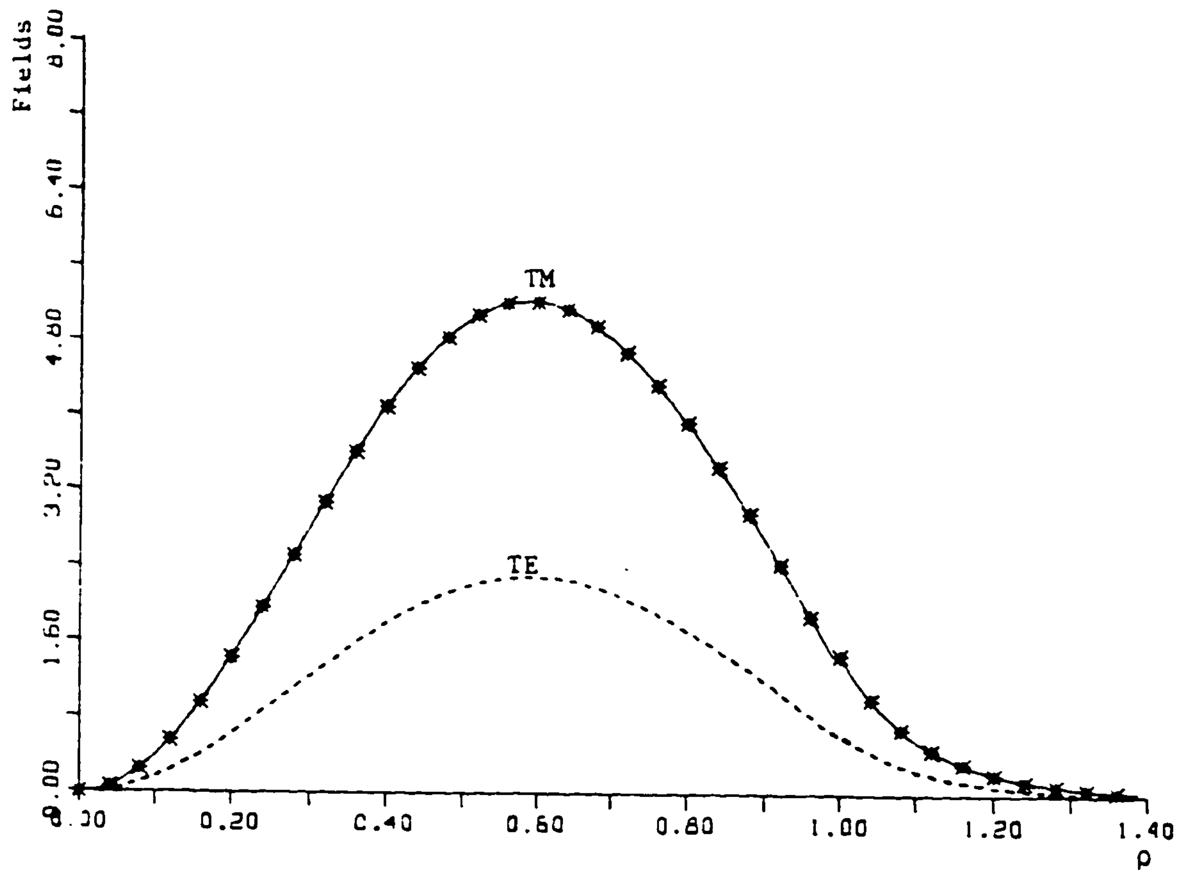
(e)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50216

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	--->	0.0	0.0	0.0	-0.0236158
L(1)	--->	0.0	0.0	0.0	0.0
L(2)	--->	0.0	0.0	-0.7418909	0.5504021
L(3)	--->	0.0	0.0	0.0	0.0
L(4)	--->	0.0	0.0	8.6454391	-3.2866964
L(5)	--->	0.0	0.0	0.0	0.0
L(6)	--->	0.0	0.0	-34.4171600	9.1305304
L(7)	--->	0.0	0.0	0.0	0.0
L(8)	--->	0.0	0.0	71.7086623	-15.1489944
L(9)	--->	0.0	0.0	0.0	0.0
L(10)	--->	0.0	0.0	-95.1810608	17.2046661
L(11)	--->	0.0	0.0	0.0	0.0
L(12)	--->	0.0	0.0	90.0807037	-14.5472164
L(13)	--->	0.0	0.0	0.0	0.0
L(14)	--->	0.0	0.0	-65.2857666	9.6845961
L(15)	--->	0.0	0.0	0.0	0.0
L(16)	--->	0.0	0.0	38.0301819	-5.2804193
L(17)	--->	0.0	0.0	0.0	0.0
L(18)	--->	0.0	0.0	-18.4315796	2.4273806
L(19)	--->	0.0	0.0	0.0	0.0
L(20)	--->	0.0	0.0	7.6256132	-0.9618096
L(21)	--->	0.0	0.0	0.0	0.0
L(22)	--->	0.0	0.0	-2.7468395	0.3342279
L(23)	--->	0.0	0.0	0.0	0.0
L(24)	--->	0.0	0.0	0.8749802	-0.1032835
L(25)	--->	0.0	0.0	0.0	0.0
L(26)	--->	0.0	0.0	-0.2495883	0.0287069
L(27)	--->	0.0	0.0	0.0	0.0
L(28)	--->	0.0	0.0	0.0644162	-0.0072445
L(29)	--->	0.0	0.0	0.0	0.0
L(30)	--->	0.0	0.0	-0.0151724	0.0016732

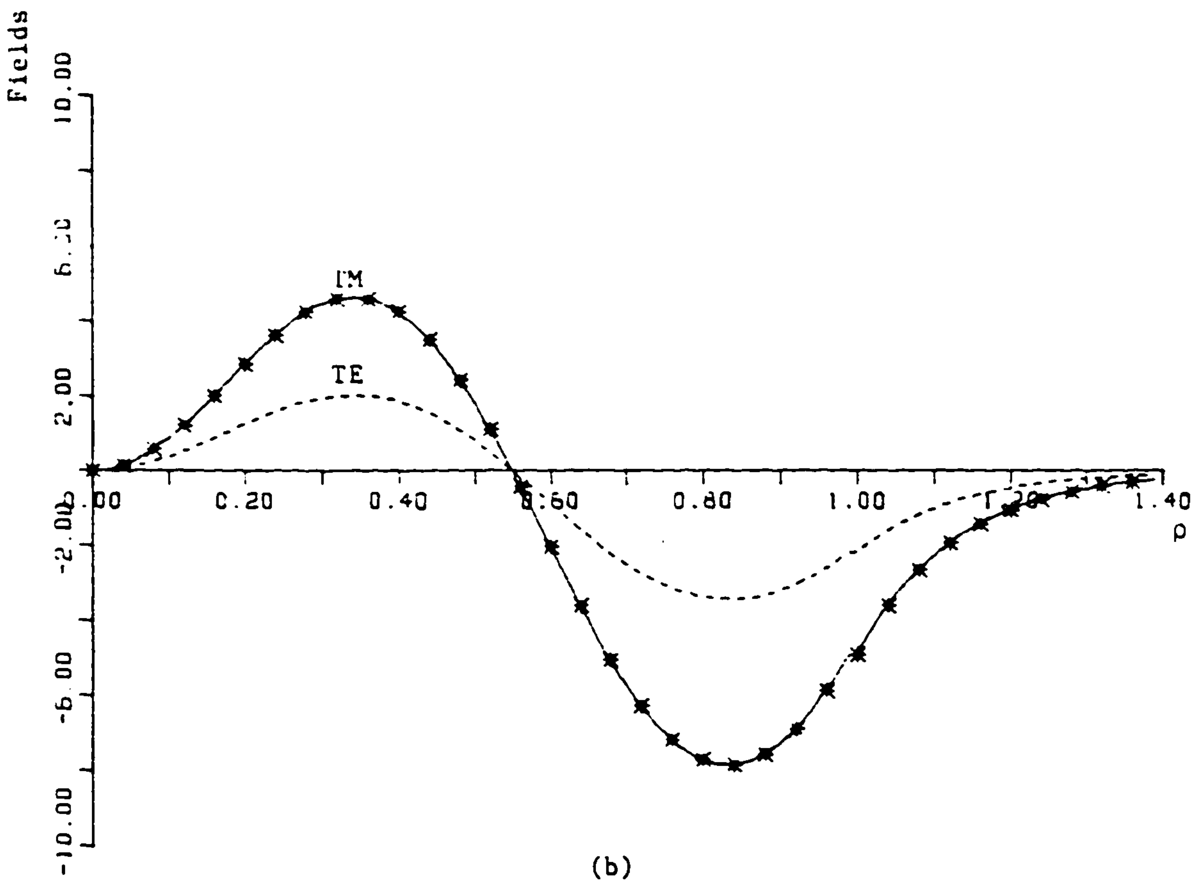
(f)



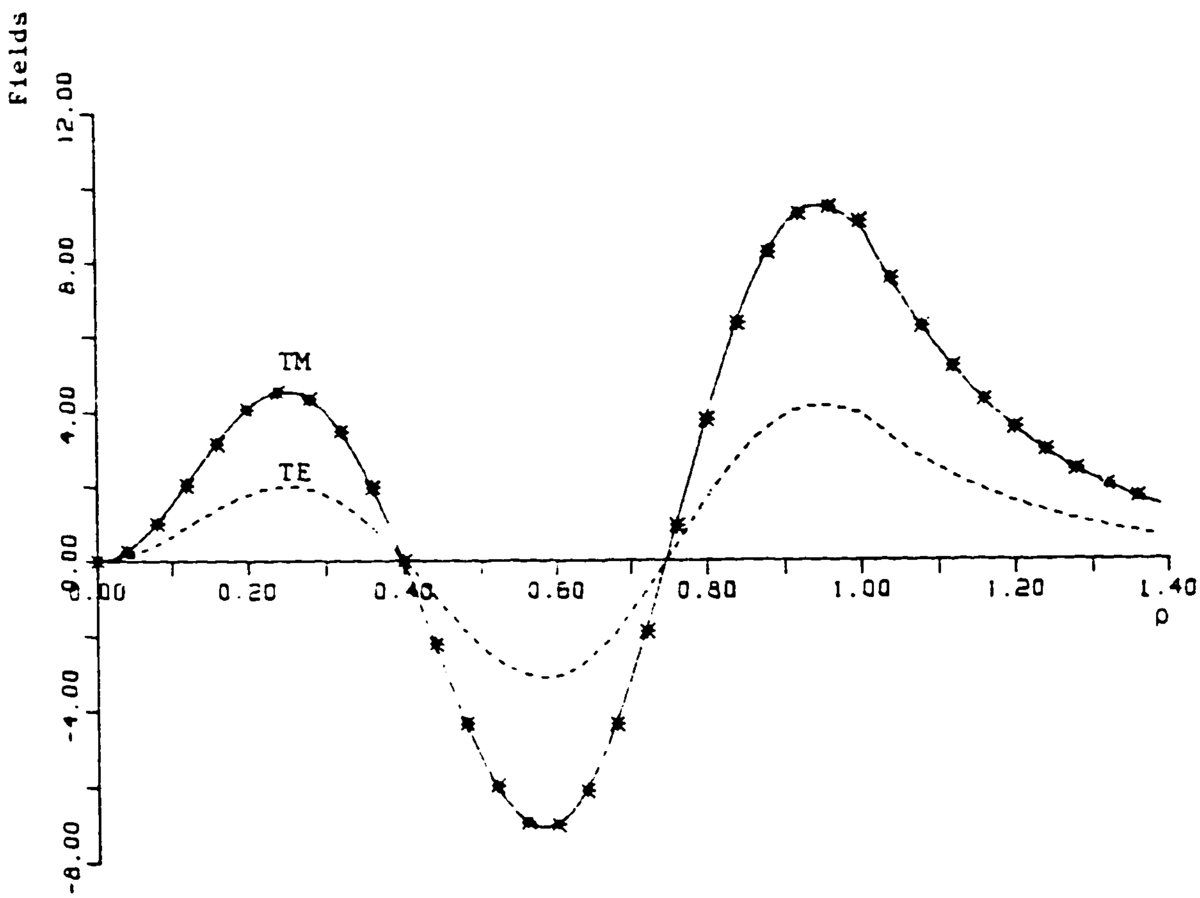
(a)

Figure 8. Field distributions for the Unperturbed case.

First Mode for p = 0



Second mode for p = 0.



(c)

Third mode for p = 0.

TABLE VIII. Coefficients in the Series Expansion for Linear Perturbation.

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50804

D1 = 0.0003 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	---->	-0.0059147	0.0	0.0	0.0
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0345253	-0.4236650	0.0	0.0
L(3)	---->	-0.0026596	0.0000847	0.0	0.0
L(4)	---->	-0.0703207	1.2371464	0.0	0.0
L(5)	---->	0.0071485	-0.0764984	0.0	0.0
L(6)	---->	0.0841995	-1.6814613	0.0	0.0
L(7)	---->	-0.0099906	0.1468922	0.0	0.0
L(8)	---->	-0.0699636	1.5115280	0.0	0.0
L(9)	---->	0.0091192	-0.1597705	0.0	0.0
L(10)	---->	0.0445105	-1.0058641	0.0	0.0
L(11)	---->	-0.0062193	0.1194274	0.0	0.0
L(12)	---->	-0.0226222	0.5338485	0.0	0.0
L(13)	---->	0.0033640	-0.0689843	0.0	0.0
L(14)	---->	0.0098106	-0.2348815	0.0	0.0
L(15)	---->	-0.0015107	0.0323719	0.0	0.0
L(16)	---->	-0.0036228	0.0884450	0.0	0.0
L(17)	---->	0.0005789	-0.0128408	0.0	0.0
L(18)	---->	0.0011735	-0.0290636	0.0	0.0
L(19)	---->	-0.0001936	0.0044073	0.0	0.0
L(20)	---->	-0.0003382	0.0084824	0.0	0.0
L(21)	---->	0.0000574	-0.0013351	0.0	0.0
L(22)	---->	0.0000878	-0.0022247	0.0	0.0
L(23)	---->	-0.0000153	0.0003618	0.0	0.0
L(24)	---->	-0.0000207	0.0005302	0.0	0.0
L(25)	---->	0.0000037	-0.0000888	0.0	0.0
L(26)	---->	0.0000045	-0.0001157	0.0	0.0
L(27)	---->	-0.0000008	0.0000199	0.0	0.0
L(28)	---->	-0.0000009	0.0000233	0.0	0.0
L(29)	---->	0.0000002	-0.0000041	0.0	0.0
L(30)	---->	0.0000002	-0.0000044	0.0	0.0

(a)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50804

D1 = 0.0003 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	---->	0.0	0.0	0.0	-0.0059147
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0	0.0	-0.1858099	0.0345253
L(3)	---->	0.0	0.0	0.0	-0.0026619
L(4)	---->	0.0	0.0	0.5423059	-0.0703470
L(5)	---->	0.0	0.0	-0.0334492	0.0071475
L(6)	---->	0.0	0.0	-0.7366499	0.0842310
L(7)	---->	0.0	0.0	0.0641536	-0.0099823
L(8)	---->	0.0	0.0	0.6615287	-0.0699592
L(9)	---->	0.0	0.0	-0.0696872	0.0091000
L(10)	---->	0.0	0.0	-0.4395534	0.0444661
L(11)	---->	0.0	0.0	0.0519773	-0.0061961
L(12)	---->	0.0	0.0	0.2328167	-0.0227662
L(13)	---->	0.0	0.0	-0.0299460	0.0033441
L(14)	---->	0.0	0.0	-0.1021715	0.0097672
L(15)	---->	0.0	0.0	0.0140072	-0.0014978
L(16)	---->	0.0	0.0	0.0383546	-0.0035976
L(17)	---->	0.0	0.0	-0.0055356	0.0005721
L(18)	---->	0.0	0.0	-0.0125577	0.0011618
L(19)	---->	0.0	0.0	0.0018918	-0.0001906
L(20)	---->	0.0	0.0	0.0036498	-0.0003336
L(21)	---->	0.0	0.0	-0.0005703	0.0000563
L(22)	---->	0.0	0.0	-0.0009527	0.0000863
L(23)	---->	0.0	0.0	0.0001537	-0.0000149
L(24)	---->	0.0	0.0	0.0002259	-0.0000203
L(25)	---->	0.0	0.0	-0.0000375	0.0000036
L(26)	---->	0.0	0.0	-0.0000490	0.0000044
L(27)	---->	0.0	0.0	0.0000084	-0.0000008
L(28)	---->	0.0	0.0	0.0000098	-0.0000009
L(29)	---->	0.0	0.0	-0.0000017	0.0000002
L(30)	---->	0.0	0.0	-0.0000018	0.0000002

(b)

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50556

D1 = 0.0003 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	——>	-0.0133896	0.0	0.0	0.0
L(1)	——>	0.0	0.0	0.0	0.0
L(2)	——>	0.1769326	-0.9590869	0.0	0.0
L(3)	——>	-0.0060141	0.0001918	0.0	0.0
L(4)	——>	-0.6295667	6.3382187	0.0	0.0
L(5)	——>	0.0365884	-0.1738362	0.0	0.0
L(6)	——>	1.1225367	-15.0444345	0.0	0.0
L(7)	——>	-0.0846173	0.7530353	0.0	0.0
L(8)	——>	-1.2792587	20.1352539	0.0	0.0
L(9)	——>	0.1131094	-1.3540134	0.0	0.0
L(10)	——>	1.0527048	-18.3743439	0.0	0.0
L(11)	——>	-0.1040541	1.4813833	0.0	0.0
L(12)	——>	-0.6724099	12.6128273	0.0	0.0
L(13)	——>	0.0722821	-1.1538839	0.0	0.0
L(14)	——>	0.3494489	-6.9126005	0.0	0.0
L(15)	——>	-0.0401511	0.6952432	0.0	0.0
L(16)	——>	-0.1527056	3.1467142	0.0	0.0
L(17)	——>	0.0185341	-0.3410629	0.0	0.0
L(18)	——>	0.0574839	-1.2236013	0.0	0.0
L(19)	——>	-0.0073085	0.1409992	0.0	0.0
L(20)	——>	-0.0189676	0.4149949	0.0	0.0
L(21)	——>	0.0025131	-0.0503540	0.0	0.0
L(22)	——>	0.0055840	-0.1247528	0.0	0.0
L(23)	——>	-0.0007657	0.0158254	0.0	0.0
L(24)	——>	-0.0014792	0.0336671	0.0	0.0
L(25)	——>	0.0002094	-0.0044408	0.0	0.0
L(26)	——>	0.0003564	-0.0082416	0.0	0.0
L(27)	——>	-0.0000519	0.0011255	0.0	0.0
L(28)	——>	-0.0000787	0.0018458	0.0	0.0
L(29)	——>	0.0000118	-0.0002601	0.0	0.0
L(30)	——>	0.0000161	-0.0003810	0.0	0.0

(c)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50556

D1 = 0.0003 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	---->	0.0	0.0	0.0	-0.0133896
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0	0.0	-0.4206338	0.1769328
L(3)	---->	0.0	0.0	0.0	-0.0060259
L(4)	---->	0.0	0.0	2.7791700	-0.6297010
L(5)	---->	0.0	0.0	-0.0757219	0.0366289
L(6)	---->	0.0	0.0	-6.5940151	1.1230822
L(7)	---->	0.0	0.0	0.3287700	-0.0846459
L(8)	---->	0.0	0.0	8.8204041	-1.2800169
L(9)	---->	0.0	0.0	-0.5909224	0.1130556
L(10)	---->	0.0	0.0	-8.0423393	1.0531149
L(11)	---->	0.0	0.0	0.6457525	-0.1038958
L(12)	---->	0.0	0.0	5.5139303	-0.6722782
L(13)	---->	0.0	0.0	-0.5021364	0.0720719
L(14)	---->	0.0	0.0	-3.0170860	0.3490248
L(15)	---->	0.0	0.0	0.3018851	-0.0399626
L(16)	---->	0.0	0.0	1.3705759	-0.1522948
L(17)	---->	0.0	0.0	-0.1476970	0.0184059
L(18)	---->	0.0	0.0	-0.5315931	0.0572169
L(19)	---->	0.0	0.0	0.0608652	-0.0072383
L(20)	---->	0.0	0.0	0.1797468	-0.0188533
L(21)	---->	0.0	0.0	-0.0216562	0.0024811
L(22)	---->	0.0	0.0	-0.0538431	0.0055280
L(23)	---->	0.0	0.0	0.0067776	-0.0007532
L(24)	---->	0.0	0.0	0.0144719	-0.0014593
L(25)	---->	0.0	0.0	-0.0018929	0.0002051
L(26)	---->	0.0	0.0	-0.0035265	0.0003502
L(27)	---->	0.0	0.0	0.0004773	-0.0000506
L(28)	---->	0.0	0.0	0.0007858	-0.0000770
L(29)	---->	0.0	0.0	-0.0001097	0.0000114
L(30)	---->	0.0	0.0	-0.0001613	0.0000156

(d)

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50207

D1 = 0.0003 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	---->	-0.0238848	0.0	0.0	0.0
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.5630099	-1.7108498	0.0	0.0
L(3)	---->	-0.0107117	0.0003422	0.0	0.0
L(4)	---->	-3.3980007	20.1665649	0.0	0.0
L(5)	---->	0.1162765	-0.3117484	0.0	0.0
L(6)	---->	9.5300474	-81.1723328	0.0	0.0
L(7)	---->	-0.4503573	2.4011745	0.0	0.0
L(8)	---->	-15.9419441	170.839569	0.0	0.0
L(9)	---->	0.9317786	-7.2196779	0.0	0.0
L(10)	---->	18.2310638	-228.789627	0.0	0.0
L(11)	---->	-1.2412062	12.2149048	0.0	0.0
L(12)	---->	-15.5054569	218.213989	0.0	0.0
L(13)	---->	1.1854266	-13.7690344	0.0	0.0
L(14)	---->	10.3739948	-159.219360	0.0	0.0
L(15)	---->	-0.8694591	11.4015112	0.0	0.0
L(16)	---->	-5.6806221	93.2985687	0.0	0.0
L(17)	---->	0.5132948	-7.3831062	0.0	0.0
L(18)	---->	2.6212111	-45.4566040	0.0	0.0
L(19)	---->	-0.2522909	3.9027195	0.0	0.0
L(20)	---->	-1.0421276	18.8964844	0.0	0.0
L(21)	---->	0.1058841	-1.7369556	0.0	0.0
L(22)	---->	0.3632644	-6.8367929	0.0	0.0
L(23)	---->	-0.0386930	0.6661752	0.0	0.0
L(24)	---->	-0.1125849	2.1868353	0.0	0.0
L(25)	---->	0.0125032	-0.2241687	0.0	0.0
L(26)	---->	0.0313804	-0.6262770	0.0	0.0
L(27)	---->	-0.0036177	0.0671353	0.0	0.0
L(28)	---->	-0.0079412	0.1622628	0.0	0.0
L(29)	---->	0.0009470	-0.0181031	0.0	0.0
L(30)	---->	0.0018393	-0.0383658	0.0	0.0

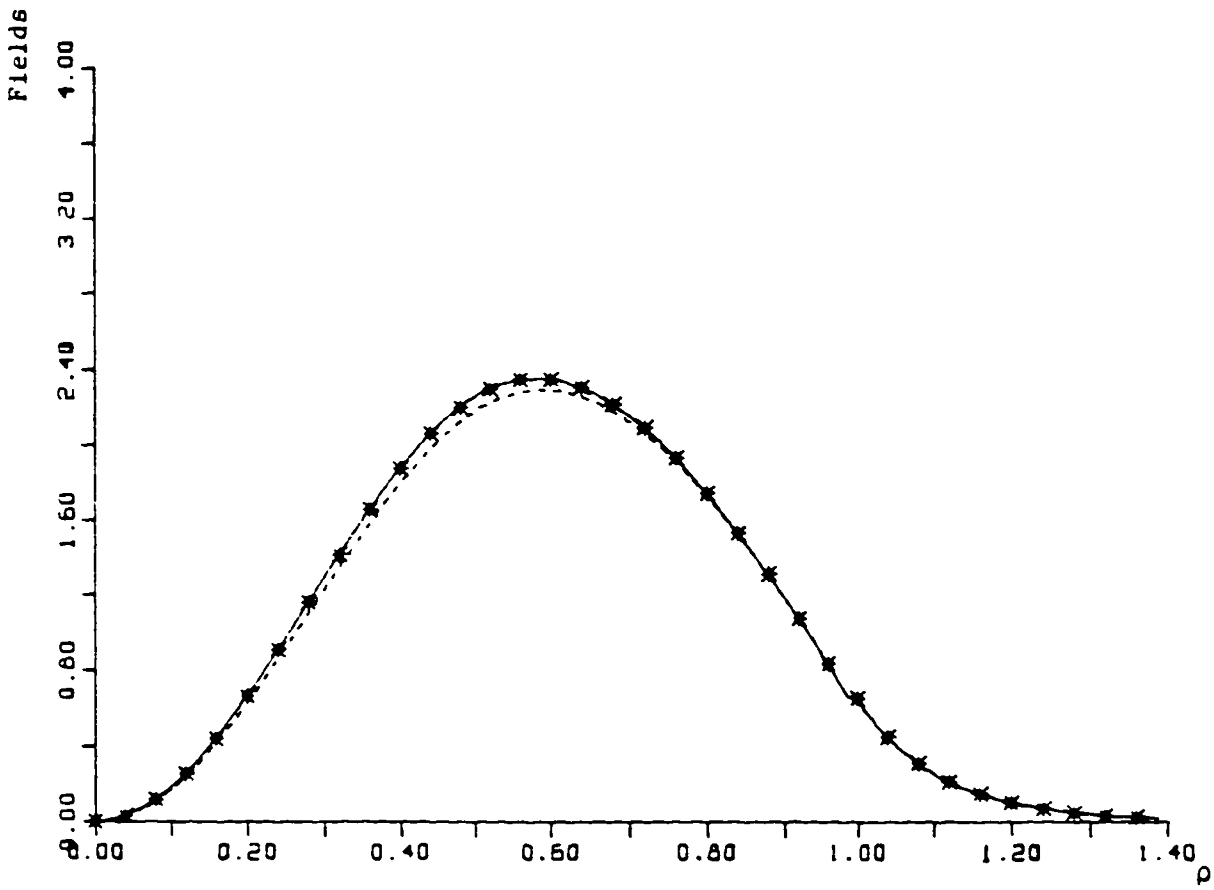
(e)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50207

D1 = 0.0003 D2 = 0.0030 D3 = 0.0 D4 = 0.0

L(0)	---->	0.0	0.0	0.0	-0.0238848
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0	0.0	-0.7503400	0.5630100
L(3)	---->	0.0	0.0	0.0	-0.0107493
L(4)	---->	0.0	0.0	8.8434753	-3.3984251
L(5)	---->	0.0	0.0	-0.1350752	0.1165551
L(6)	---->	0.0	0.0	-35.5871429	9.5333071
L(7)	---->	0.0	0.0	1.0461645	-0.4510654
L(8)	---->	0.0	0.0	74.8721924	-15.9510899
L(9)	---->	0.0	0.0	-3.1489353	0.9325538
L(10)	---->	0.0	0.0	-100.220612	18.2442017
L(11)	---->	0.0	0.0	5.3265734	-1.2412777
L(12)	---->	0.0	0.0	95.5235291	-15.5161428
L(13)	---->	0.0	0.0	-5.9991884	1.1844196
L(14)	---->	0.0	0.0	-69.6341705	10.3781757
L(15)	---->	0.0	0.0	4.9611368	-0.8677392
L(16)	---->	0.0	0.0	40.7537842	-5.6795149
L(17)	---->	0.0	0.0	-3.2070599	0.5115556
L(18)	---->	0.0	0.0	-19.8246460	2.6181993
L(19)	---->	0.0	0.0	1.6916323	-0.2509962
L(20)	---->	0.0	0.0	8.2250700	-1.0395317
L(21)	---->	0.0	0.0	-0.7509565	0.1051172
L(22)	---->	0.0	0.0	-2.9688063	0.3617222
L(23)	---->	0.0	0.0	0.2871526	-0.0383156
L(24)	---->	0.0	0.0	0.9469578	-0.1118608
L(25)	---->	0.0	0.0	-0.0962948	0.0123447
L(26)	---->	0.0	0.0	-0.2703158	0.0310961
L(27)	---->	0.0	0.0	0.0287265	-0.0035597
L(28)	---->	0.0	0.0	0.0697774	-0.0078448
L(29)	---->	0.0	0.0	-0.0077123	0.0009282
L(30)	---->	0.0	0.0	-0.0164296	0.0018104

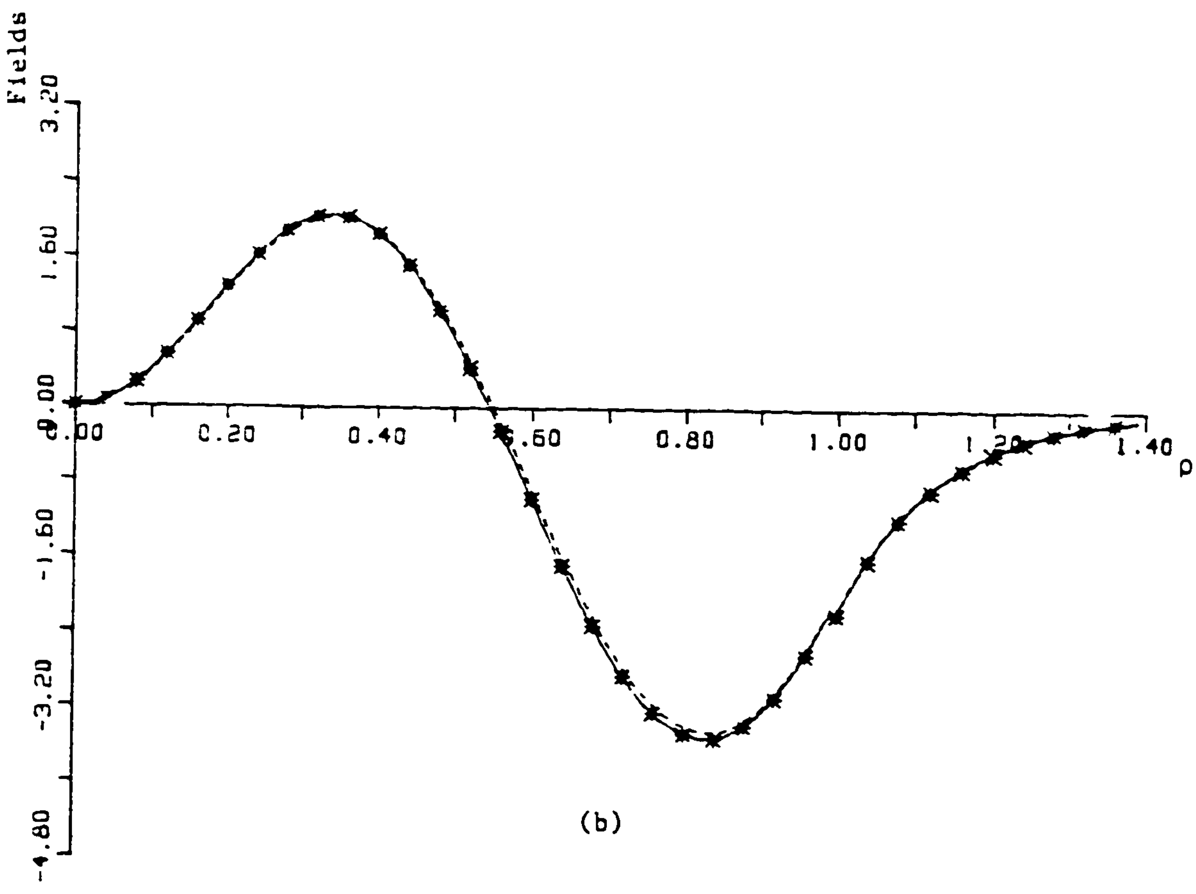
(f)



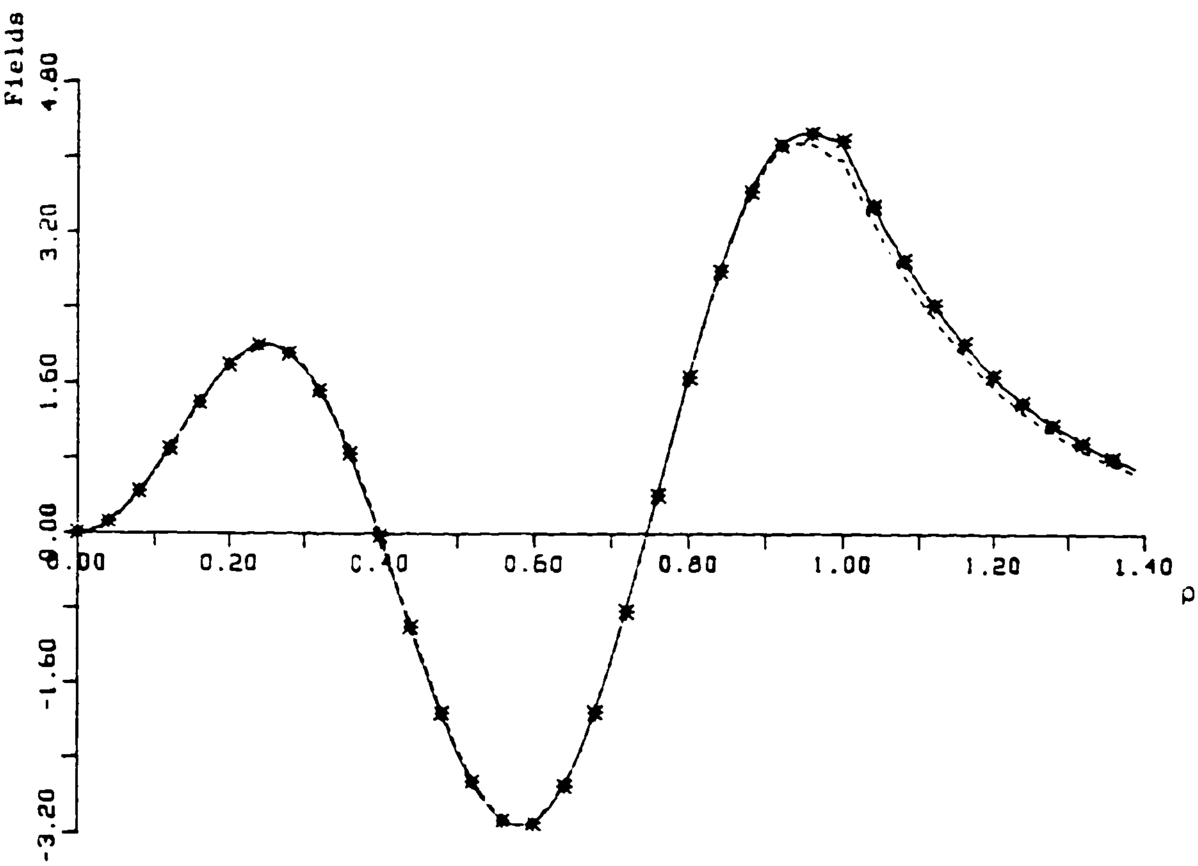
(a)

Figure 9. Field distributions for Linear Perturbation

First Mode for p = 1



Second mode for p = 1



(c)

Third mode for p = 1

TABLE IX. Coefficients in the Series Expansion for Cubic Perturbation.

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50802

D1 = 0.0 D2 = 0.0030 D3 = 0.0010 D4 = 0.0

L(0)	---->	-0.0059748	0.0	0.0	0.0
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0352303	-0.4279686	0.0	0.0
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	-0.0720744	1.2624016	0.0	0.0
L(5)	---->	-0.0053694	0.0001712	0.0	0.0
L(6)	---->	0.0868444	-1.7234030	0.0	0.0
L(7)	---->	0.0139073	-0.1106083	0.0	0.0
L(8)	---->	-0.0725823	1.5590229	0.0	0.0
L(9)	---->	-0.0163854	0.2227751	0.0	0.0
L(10)	---->	0.0457953	-1.0434589	0.0	0.0
L(11)	---->	0.0162977	-0.2411170	0.0	0.0
L(12)	---->	-0.0227181	0.5491478	0.0	0.0
L(13)	---->	-0.0108091	0.1610066	0.0	0.0
L(14)	---->	0.0090430	-0.2336862	0.0	0.0
L(15)	---->	0.0057057	-0.1041374	0.0	0.0
L(16)	---->	-0.0028748	0.0814320	0.0	0.0
L(17)	---->	-0.0024832	0.0485468	0.0	0.0
L(18)	---->	0.0006984	-0.0230096	0.0	0.0
L(19)	---->	0.0009129	-0.0189203	0.0	0.0
L(20)	---->	-0.0001055	0.0050236	0.0	0.0
L(21)	---->	-0.0002874	0.0062979	0.0	0.0
L(22)	---->	-0.0000058	-0.0006847	0.0	0.0
L(23)	---->	0.0000781	-0.0018116	0.0	0.0
L(24)	---->	0.0000113	-0.0000384	0.0	0.0
L(25)	---->	-0.0000183	0.0004533	0.0	0.0
L(26)	---->	-0.0000051	0.0000642	0.0	0.0
L(27)	---->	0.0000037	-0.0000984	0.0	0.0
L(28)	---->	0.0000016	-0.0000264	0.0	0.0
L(29)	---->	-0.0000006	0.0000183	0.0	0.0
L(30)	---->	-0.0000004	0.0000078	0.0	0.0

(a)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50802

D1 = 0.0 D2 = 0.0030 D3 = 0.0010 D4 = 0.0

L(0)	---->	0.0	0.0	0.0	-0.0059748
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0	0.0	-0.1876973	0.0352303
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	0.0	0.0	0.5533793	-0.0721008
L(5)	---->	0.0	0.0	0.0	-0.0053778
L(6)	---->	0.0	0.0	-0.7550153	0.0868763
L(7)	---->	0.0	0.0	-0.0482700	0.0139138
L(8)	---->	0.0	0.0	0.6823044	-0.0725775
L(9)	---->	0.0	0.0	0.0971337	-0.0183746
L(10)	---->	0.0	0.0	-0.4560043	0.0457504
L(11)	---->	0.0	0.0	-0.1049520	0.0162627
L(12)	---->	0.0	0.0	0.2395415	-0.0226656
L(13)	---->	0.0	0.0	0.0785989	-0.0107644
L(14)	---->	0.0	0.0	-0.1017197	0.0090071
L(15)	---->	0.0	0.0	-0.0450887	0.0056681
L(16)	---->	0.0	0.0	0.0353696	-0.0028584
L(17)	---->	0.0	0.0	0.0209488	-0.0024597
L(18)	---->	0.0	0.0	-0.0099774	0.0006937
L(19)	---->	0.0	0.0	-0.0081337	0.0009012
L(20)	---->	0.0	0.0	0.0021793	-0.0001053
L(21)	---->	0.0	0.0	0.0026964	-0.0002827
L(22)	---->	0.0	0.0	-0.0003006	-0.0000053
L(23)	---->	0.0	0.0	-0.0007723	0.0000766
L(24)	---->	0.0	0.0	-0.0000138	0.0000109
L(25)	---->	0.0	0.0	0.0001924	-0.0000179
L(26)	---->	0.0	0.0	0.0000264	-0.0000049
L(27)	---->	0.0	0.0	-0.0000416	0.0000036
L(28)	---->	0.0	0.0	-0.0000109	0.0000015
L(29)	---->	0.0	0.0	0.0000077	-0.0000006
L(30)	---->	0.0	0.0	0.0000032	-0.0000004

(b)

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50546

D1 = 0.0 D2 = 0.0030 D3 = 0.0010 D4 = 0.0

L(0)	----->	-0.0136909	0.0	0.0	0.0
L(1)	----->	0.0	0.0	0.0	0.0
L(2)	----->	0.1849867	-0.9806728	0.0	0.0
L(3)	----->	0.0	0.0	0.0	0.0
L(4)	----->	-0.6709399	6.6267014	0.0	0.0
L(5)	----->	-0.0122787	0.0003923	0.0	0.0
L(6)	----->	1.2150354	-16.0328979	0.0	0.0
L(7)	----->	0.0728800	-0.2550762	0.0	0.0
L(8)	----->	-1.4032507	21.7940826	0.0	0.0
L(9)	----->	-0.1656659	1.1713419	0.0	0.0
L(10)	----->	1.1671066	-20.1548004	0.0	0.0
L(11)	----->	0.2183244	-2.1762199	0.0	0.0
L(12)	----->	-0.7481783	13.9825554	0.0	0.0
L(13)	----->	-0.1986167	2.4268513	0.0	0.0
L(14)	----->	0.3646444	-7.6900556	0.0	0.0
L(15)	----->	0.1366758	-1.9143038	0.0	0.0
L(16)	----->	-0.1622104	3.4621191	0.0	0.0
L(17)	----->	-0.0751598	1.1630888	0.0	0.0
L(18)	----->	0.0566122	-1.2966040	0.0	0.0
L(19)	----->	0.0342205	-0.5726915	0.0	0.0
L(20)	----->	-0.0162298	0.4080162	0.0	0.0
L(21)	----->	-0.0132128	0.2360912	0.0	0.0
L(22)	----->	0.0036719	-0.1063010	0.0	0.0
L(23)	----->	0.0043973	-0.0832896	0.0	0.0
L(24)	----->	-0.0005614	0.0220046	0.0	0.0
L(25)	----->	-0.0012748	0.0255181	0.0	0.0
L(26)	----->	0.0000032	-0.0030811	0.0	0.0
L(27)	----->	0.0003237	-0.0068536	0.0	0.0
L(28)	----->	0.0000363	0.0000025	0.0	0.0
L(29)	----->	-0.0000720	0.0016209	0.0	0.0
L(30)	----->	-0.0000165	0.0001792	0.0	0.0

(c)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50546

D1 = 0.0 D2 = 0.0030 D3 = 0.0010 D4 = 0.0

L(0)	---->	0.0	0.0	0.0	-0.0136909
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0	0.0	-0.4301010	0.1849869
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	0.0	0.0	2.9056807	-0.6710799
L(5)	---->	0.0	0.0	0.0	-0.0123231
L(6)	---->	0.0	0.0	-7.0273190	1.2156162
L(7)	---->	0.0	0.0	-0.1106088	0.0730585
L(8)	---->	0.0	0.0	9.5471420	-1.4040794
L(9)	---->	0.0	0.0	0.5100292	-0.1658875
L(10)	---->	0.0	0.0	-8.8216241	1.1675835
L(11)	---->	0.0	0.0	-0.9475189	0.2183603
L(12)	---->	0.0	0.0	6.1132689	-0.7480619
L(13)	---->	0.0	0.0	1.0553513	-0.1983720
L(14)	---->	0.0	0.0	-3.3572817	0.3842523
L(15)	---->	0.0	0.0	-0.8309139	0.1362730
L(16)	---->	0.0	0.0	1.5089102	-0.1618535
L(17)	---->	0.0	0.0	0.5036486	-0.0747827
L(18)	---->	0.0	0.0	-0.5649584	0.0564131
L(19)	---->	0.0	0.0	-0.2472944	0.0339654
L(20)	---->	0.0	0.0	0.1772217	-0.0161550
L(21)	---->	0.0	0.0	0.1016213	-0.0130777
L(22)	---->	0.0	0.0	-0.0461372	0.0036557
L(23)	---->	0.0	0.0	-0.0357248	0.0043388
L(24)	---->	0.0	0.0	0.0095704	-0.0005626
L(25)	---->	0.0	0.0	0.0109044	-0.0012537
L(26)	---->	0.0	0.0	-0.0013596	0.0000063
L(27)	---->	0.0	0.0	-0.0029175	0.0003173
L(28)	---->	0.0	0.0	0.0000142	0.0000345
L(29)	---->	0.0	0.0	0.0006874	-0.0000703
L(30)	---->	0.0	0.0	0.0000722	-0.0000158

(d)

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50180

D1 = 0.0 D2 = 0.0030 D3 = 0.0010 D4 = 0.0

L(0)	---->	-0.0246983	0.0	0.0	0.0
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.6020135	-1.7691193	0.0	0.0
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	-3.7513914	21.5635529	0.0	0.0
L(5)	---->	-0.0220862	0.0007076	0.0	0.0
L(6)	---->	10.8348751	-89.6129303	0.0	0.0
L(7)	---->	0.2366147	-0.4643265	0.0	0.0
L(8)	---->	-18.6117096	194.225159	0.0	0.0
L(9)	---->	-0.9178459	3.8271103	0.0	0.0
L(10)	---->	21.7962646	-267.093994	0.0	0.0
L(11)	---->	1.9111090	-12.1039248	0.0	0.0
L(12)	---->	-18.9254608	260.872070	0.0	0.0
L(13)	---->	-2.5653715	21.2956238	0.0	0.0
L(14)	---->	12.8669052	-194.319290	0.0	0.0
L(15)	---->	2.4691467	-24.7636566	0.0	0.0
L(16)	---->	-7.1030064	115.697174	0.0	0.0
L(17)	---->	-1.8239851	21.0317535	0.0	0.0
L(18)	---->	3.2612123	-56.8182678	0.0	0.0
L(19)	---->	1.0829306	-13.9055672	0.0	0.0
L(20)	---->	-1.2638302	23.4947662	0.0	0.0
L(21)	---->	-0.5338144	7.4733610	0.0	0.0
L(22)	---->	C.4160277	-8.2815838	0.0	0.0
L(23)	---->	0.2236546	-3.3654823	0.0	0.0
L(24)	---->	-0.1158631	2.4994802	0.0	0.0
L(25)	---->	-0.0810239	1.2980356	0.0	0.0
L(26)	---->	0.0266931	-0.6423345	0.0	0.0
L(27)	---->	0.0256987	-0.4356702	0.0	0.0
L(28)	---->	-0.0047311	0.1372061	0.0	0.0
L(29)	---->	-0.0071985	0.1287235	0.0	0.0
L(30)	---->	0.0004626	-0.0225876	0.0	0.0

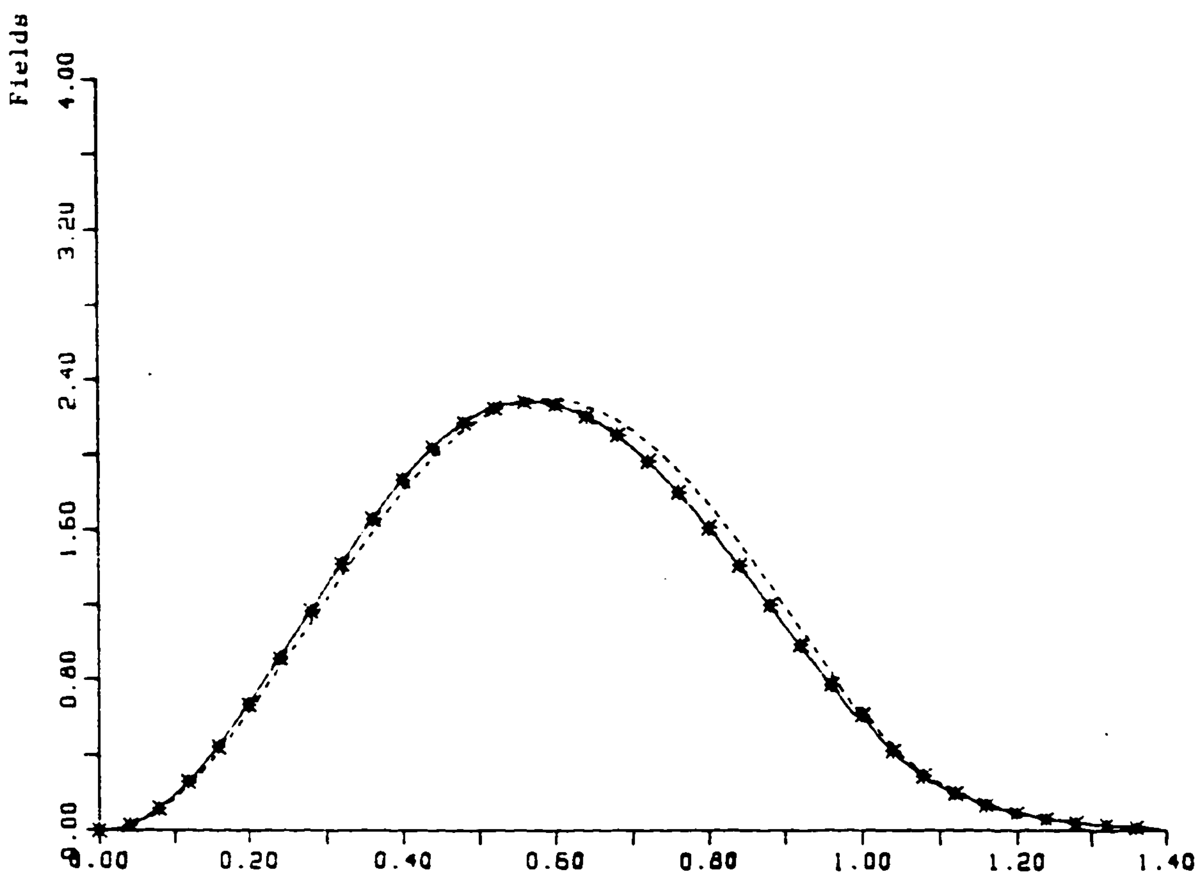
(e)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50180

D1 = 0.0 D2 = 0.0030 D3 = 0.0010 D4 = 0.0

L(0)	---->	0.0	0.0	0.0	-0.0246983
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0	0.0	-0.7758951	0.6020132
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	0.0	0.0	9.4561195	-3.7518425
L(5)	---->	0.0	0.0	0.0	-0.0222307
L(6)	---->	0.0	0.0	-39.2880249	10.8384657
L(7)	---->	0.0	0.0	-0.1995364	0.2377583
L(8)	---->	0.0	0.0	85.1225586	-18.6221619
L(9)	---->	0.0	0.0	1.6598158	-0.9210874
L(10)	---->	0.0	0.0	-117.002960	21.8118896
L(11)	---->	0.0	0.0	-5.2610798	1.9156332
L(12)	---->	0.0	0.0	114.203354	-18.9389038
L(13)	---->	0.0	0.0	9.2584000	-2.5684490
L(14)	---->	0.0	0.0	-84.9950104	12.8730412
L(15)	---->	0.0	0.0	-10.7583733	2.4689569
L(16)	---->	0.0	0.0	50.5508118	-7.1029177
L(17)	---->	0.0	0.0	9.1249714	-1.8211746
L(18)	---->	0.0	0.0	-24.7931061	3.2587767
L(19)	---->	0.0	0.0	-6.0223341	1.0794163
L(20)	---->	0.0	0.0	10.2374430	-1.2617302
L(21)	---->	0.0	0.0	3.2295094	-0.5310284
L(22)	---->	0.0	0.0	-3.6033850	0.4149432
L(23)	---->	0.0	0.0	-1.4506302	0.2219843
L(24)	---->	0.0	0.0	1.0862856	-0.1154866
L(25)	---->	0.0	0.0	0.5578907	-0.0802156
L(26)	---->	0.0	0.0	-0.2790778	0.0266198
L(27)	---->	0.0	0.0	-0.1866645	0.0253723
L(28)	---->	0.0	0.0	0.0597328	-0.0047397
L(29)	---->	0.0	0.0	0.0549703	-0.0070864
L(30)	---->	0.0	0.0	-0.0099265	0.0004778

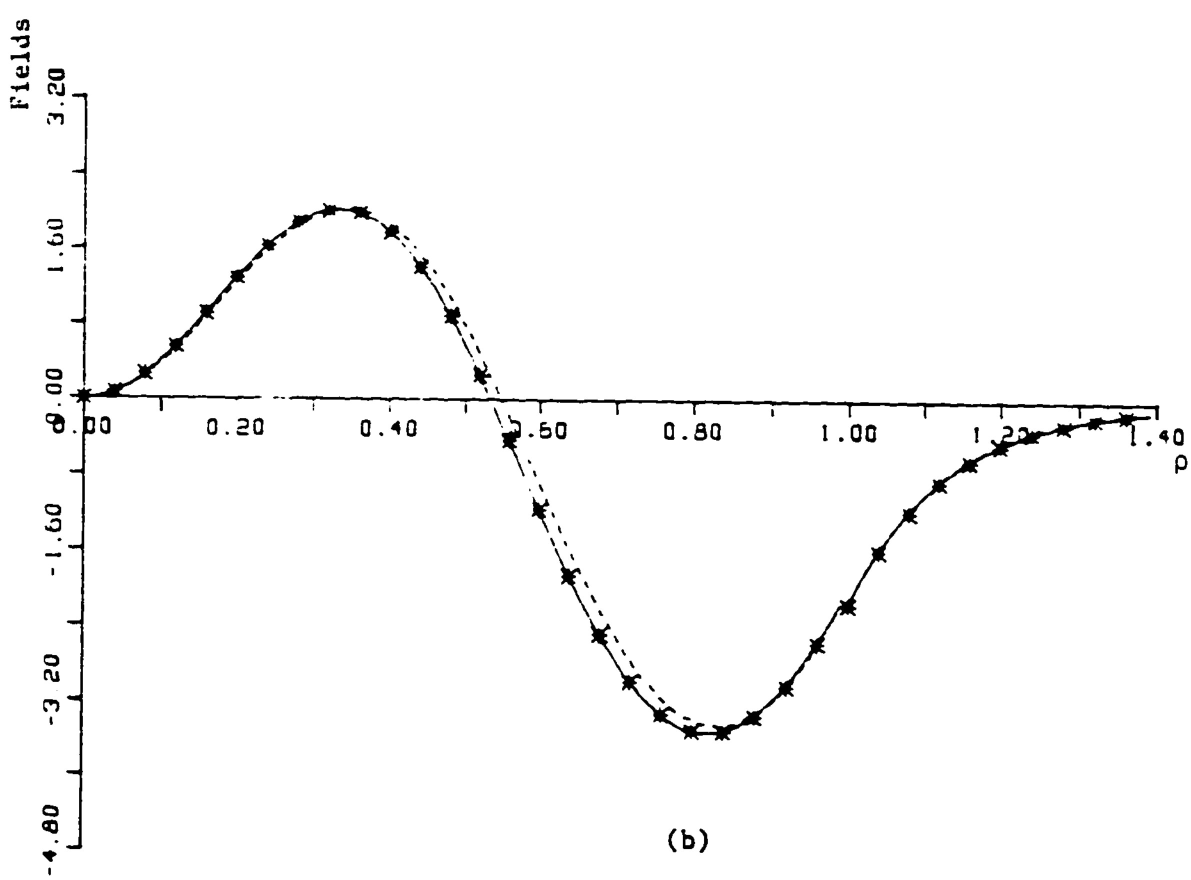
(f)



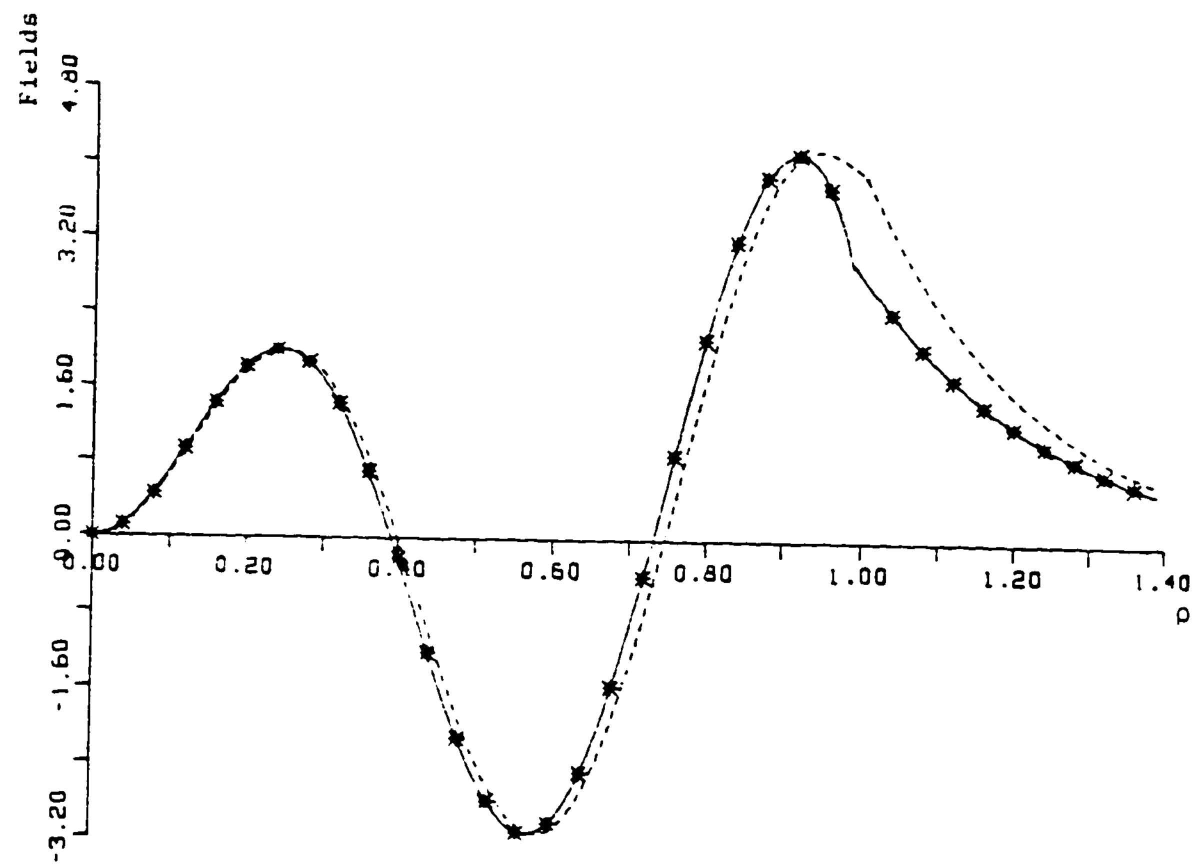
(a)

Figure 10. Field distributions for Cubic Perturbation

First mode for p = 3.



Second mode for p = 3



(c)

Third mode for p = 3.

TABLE X. Coefficients in the Series Expansion for Quartic Perturbation.

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50797

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0020

L(0)	---->	-0.0061274	0.0	0.0	0.0
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0370527	-0.4388984	0.0	0.0
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	-0.0766695	1.3276882	0.0	0.0
L(5)	---->	0.0	0.0	0.0	0.0
L(6)	---->	0.0840105	-1.8329554	0.0	0.0
L(7)	---->	0.0	0.0	0.0	0.0
L(8)	---->	-0.0540822	1.5071955	0.0	0.0
L(9)	---->	0.0	0.0	0.0	0.0
L(10)	---->	0.0184672	-0.7761875	0.0	0.0
L(11)	---->	0.0	0.0	0.0	0.0
L(12)	---->	0.0004802	0.2203968	0.0	0.0
L(13)	---->	0.0	0.0	0.0	0.0
L(14)	---->	-0.0040545	0.0054533	0.0	0.0
L(15)	---->	0.0	0.0	0.0	0.0
L(16)	---->	0.0021771	-0.0366464	0.0	0.0
L(17)	---->	0.0	0.0	0.0	0.0
L(18)	---->	-0.0005081	0.0174161	0.0	0.0
L(19)	---->	0.0	0.0	0.0	0.0
L(20)	---->	-0.0000353	-0.0036283	0.0	0.0
L(21)	---->	0.0	0.0	0.0	0.0
L(22)	---->	0.0000701	-0.0002483	0.0	0.0
L(23)	---->	0.0	0.0	0.0	0.0
L(24)	---->	-0.0000239	0.0004252	0.0	0.0
L(25)	---->	0.0	0.0	0.0	0.0
L(26)	---->	0.0000027	-0.0001324	0.0	0.0
L(27)	---->	0.0	0.0	0.0	0.0
L(28)	---->	0.0000009	0.0000137	0.0	0.0
L(29)	---->	0.0	0.0	0.0	0.0
L(30)	---->	-0.0000005	0.0000046	0.0	0.0

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50797

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0020

L(0)	----->	0.0	0.0	0.0	-0.006127
L(1)	----->	0.0	0.0	0.0	0.0
L(2)	----->	0.0	0.0	-0.1924909	0.037052
L(3)	----->	0.0	0.0	0.0	0.0
L(4)	----->	0.0	0.0	0.5820055	-0.076697
L(5)	----->	0.0	0.0	0.0	0.0
L(6)	----->	0.0	0.0	-0.8031487	0.08402
L(7)	----->	0.0	0.0	0.0	0.0
L(8)	----->	0.0	0.0	0.6599469	-0.05406
L(9)	----->	0.0	0.0	0.0	0.0
L(10)	----->	0.0	0.0	-0.3396640	0.01842
L(11)	----->	0.0	0.0	0.0	0.0
L(12)	----->	0.0	0.0	0.0964846	0.00049
L(13)	----->	0.0	0.0	0.0	0.0
L(14)	----->	0.0	0.0	0.0022297	-0.00405
L(15)	----->	0.0	0.0	0.0	0.0
L(16)	----->	0.0	0.0	-0.0159069	0.00217
L(17)	----->	0.0	0.0	0.0	0.0
L(18)	----->	0.0	0.0	0.0075760	-0.00050
L(19)	----->	0.0	0.0	0.0	0.0
L(20)	----->	0.0	0.0	-0.0015907	-0.00003
L(21)	----->	0.0	0.0	0.0	0.0
L(22)	----->	0.0	0.0	-0.0000984	0.00006
L(23)	----->	0.0	0.0	0.0	0.0
L(24)	----->	0.0	0.0	0.0001815	-0.00002
L(25)	----->	0.0	0.0	0.0	0.0
L(26)	----->	0.0	0.0	-0.0000572	0.00000
L(27)	----->	0.0	0.0	0.0	0.0
L(28)	----->	0.0	0.0	0.0000062	0.00000
L(29)	----->	0.0	0.0	0.0	0.0
L(30)	----->	0.0	0.0	0.0000019	-0.00000

(b)

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50532

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0020

L(0)	---->	-0.0141134	0.0	0.0	0.0
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.1965796	-1.0109348	0.0	0.0
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	-0.7320080	7.0419464	0.0	0.0
L(5)	---->	0.0	0.0	0.0	0.0
L(6)	---->	1.3325491	-17.4911346	0.0	0.0
L(7)	---->	0.0	0.0	0.0	0.0
L(8)	---->	-1.4612341	23.8946686	0.0	0.0
L(9)	---->	0.0	0.0	0.0	0.0
L(10)	---->	1.0446959	-20.9697266	0.0	0.0
L(11)	---->	0.0	0.0	0.0	0.0
L(12)	---->	-0.4835373	12.4923201	0.0	0.0
L(13)	---->	0.0	0.0	0.0	0.0
L(14)	---->	0.1176056	-4.9500809	0.0	0.0
L(15)	---->	0.0	0.0	0.0	0.0
L(16)	---->	0.0140943	1.0472803	0.0	0.0
L(17)	---->	0.0	0.0	0.0	0.0
L(18)	---->	-0.0259735	0.1170622	0.0	0.0
L(19)	---->	0.0	0.0	0.0	0.0
L(20)	---->	0.0112921	-0.1680339	0.0	0.0
L(21)	---->	0.0	0.0	0.0	0.0
L(22)	---->	-0.0022421	0.0738553	0.0	0.0
L(23)	---->	0.0	0.0	0.0	0.0
L(24)	---->	-0.0001668	-0.0132757	0.0	0.0
L(25)	---->	0.0	0.0	0.0	0.0
L(26)	---->	0.0002758	-0.0011169	0.0	0.0
L(27)	---->	0.0	0.0	0.0	0.0
L(28)	---->	-0.0000866	0.0014369	0.0	0.0
L(29)	---->	0.0	0.0	0.0	0.0
L(30)	---->	0.0000097	-0.0004159	0.0	0.0

(c)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50532

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0020

L(0)	---->	0.0	0.0	0.0	-0.0141134
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0	0.0	-0.4433731	0.1965797
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	0.0	0.0	3.0877743	-0.7321559
L(5)	---->	0.0	0.0	0.0	0.0
L(6)	---->	0.0	0.0	-7.6668882	1.3330994
L(7)	---->	0.0	0.0	0.0	0.0
L(8)	---->	0.0	0.0	10.4698257	-1.4618015
L(9)	---->	0.0	0.0	0.0	0.0
L(10)	---->	0.0	0.0	-9.1844921	1.0447292
L(11)	---->	0.0	0.0	0.0	0.0
L(12)	---->	0.0	0.0	5.4700251	-0.4831677
L(13)	---->	0.0	0.0	0.0	0.0
L(14)	---->	0.0	0.0	-2.1663846	0.1172984
L(15)	---->	0.0	0.0	0.0	0.0
L(16)	---->	0.0	0.0	0.4606159	0.0141805
L(17)	---->	0.0	0.0	0.0	0.0
L(18)	---->	0.0	0.0	0.0494977	-0.0259561
L(19)	---->	0.0	0.0	0.0	0.0
L(20)	---->	0.0	0.0	-0.0815410	0.0112777
L(21)	---->	0.0	0.0	0.0	0.0
L(22)	---->	0.0	0.0	0.0322060	-0.0022465
L(23)	---->	0.0	0.0	0.0	0.0
L(24)	---->	0.0	0.0	-0.0058811	-0.0001794
L(25)	---->	0.0	0.0	0.0	0.0
L(26)	---->	0.0	0.0	-0.0004335	0.0002725
L(27)	---->	0.0	0.0	0.0	0.0
L(28)	---->	0.0	0.0	0.0006115	-0.0000862
L(29)	---->	0.0	0.0	0.0	0.0
L(30)	---->	0.0	0.0	-0.0001805	0.0000100

(d)

COMPONENTS OF COLUMN VECTOR L FOR TM MODES WITH ALPHA = 1.50183

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0020

L(0)	---->	-0.0246067	0.0	0.0	0.0
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.5975587	-1.7625608	0.0	0.0
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	-3.7104445	21.4039917	0.0	0.0
L(5)	---->	0.0	0.0	0.0	0.0
L(6)	---->	10.6450939	-88.6337128	0.0	0.0
L(7)	---->	0.0	0.0	0.0	0.0
L(8)	---->	-17.9051208	190.803070	0.0	0.0
L(9)	---->	0.0	0.0	0.0	0.0
L(10)	---->	19.8722667	-256.857178	0.0	0.0
L(11)	---->	0.0	0.0	0.0	0.0
L(12)	---->	-15.4416103	237.626526	0.0	0.0
L(13)	---->	0.0	0.0	0.0	0.0
L(14)	---->	8.5486358	-158.253784	0.0	0.0
L(15)	---->	0.0	0.0	0.0	0.0
L(16)	---->	-3.2443886	76.6022949	0.0	0.0
L(17)	---->	0.0	0.0	0.0	0.0
L(18)	---->	0.6717544	-25.7797852	0.0	0.0
L(19)	---->	0.0	0.0	0.0	0.0
L(20)	---->	0.0744152	4.7589779	0.0	0.0
L(21)	---->	0.0	0.0	0.0	0.0
L(22)	---->	-0.1211593	0.5137040	0.0	0.0
L(23)	---->	0.0	0.0	0.0	0.0
L(24)	---->	0.0493359	-0.7325642	0.0	0.0
L(25)	---->	0.0	0.0	0.0	0.0
L(26)	---->	-0.0099177	0.2730210	0.0	0.0
L(27)	---->	0.0	0.0	0.0	0.0
L(28)	---->	-0.0002170	-0.0502600	0.0	0.0
L(29)	---->	0.0	0.0	0.0	0.0
L(30)	---->	0.0006497	-0.0013652	0.0	0.0

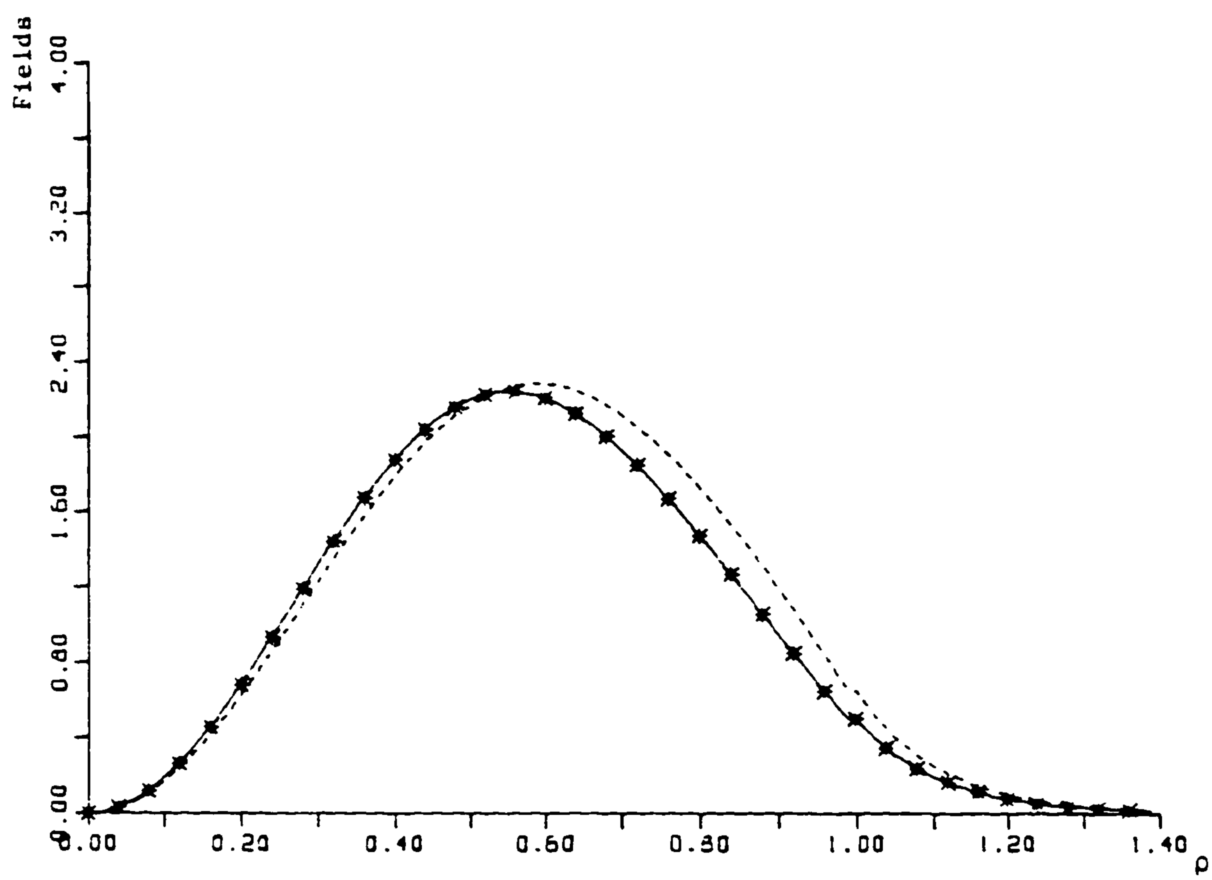
(e)

COMPONENTS OF COLUMN VECTOR L FOR TE MODES WITH ALPHA = 1.50183

D1 = 0.0 D2 = 0.0030 D3 = 0.0 D4 = 0.0020

L(0)	---->	0.0	0.0	0.0	-0.0246067
L(1)	---->	0.0	0.0	0.0	0.0
L(2)	---->	0.0	0.0	-0.7730193	0.5975589
L(3)	---->	0.0	0.0	0.0	0.0
L(4)	---->	0.0	0.0	9.3661504	-3.7108927
L(5)	---->	0.0	0.0	0.0	0.0
L(6)	---->	0.0	0.0	-38.8592072	10.6483841
L(7)	---->	0.0	0.0	0.0	0.0
L(8)	---->	0.0	0.0	83.6296997	-17.9132538
L(9)	---->	0.0	0.0	0.0	0.0
L(10)	---->	0.0	0.0	-112.548874	19.8812866
L(11)	---->	0.0	0.0	0.0	0.0
L(12)	---->	0.0	0.0	104.095047	-15.4453640
L(13)	---->	0.0	0.0	0.0	0.0
L(14)	---->	0.0	0.0	-69.3165283	8.5468855
L(15)	---->	0.0	0.0	0.0	0.0
L(16)	---->	0.0	0.0	33.5625305	-3.2409067
L(17)	---->	0.0	0.0	0.0	0.0
L(18)	---->	0.0	0.0	-11.3125610	0.6696467
L(19)	---->	0.0	0.0	0.0	0.0
L(20)	---->	0.0	0.0	2.1036940	0.0750992
L(21)	---->	0.0	0.0	0.0	0.0
L(22)	---->	0.0	0.0	0.2144765	-0.1213050
L(23)	---->	0.0	0.0	0.0	0.0
L(24)	---->	0.0	0.0	-0.3175664	0.0494121
L(25)	---->	0.0	0.0	0.0	0.0
L(26)	---->	0.0	0.0	0.1194062	-0.0099861
L(27)	---->	0.0	0.0	0.0	0.0
L(28)	---->	0.0	0.0	-0.0224080	-0.0001795
L(29)	---->	0.0	0.0	0.0	0.0
L(30)	---->	0.0	0.0	-0.0003760	0.0008388

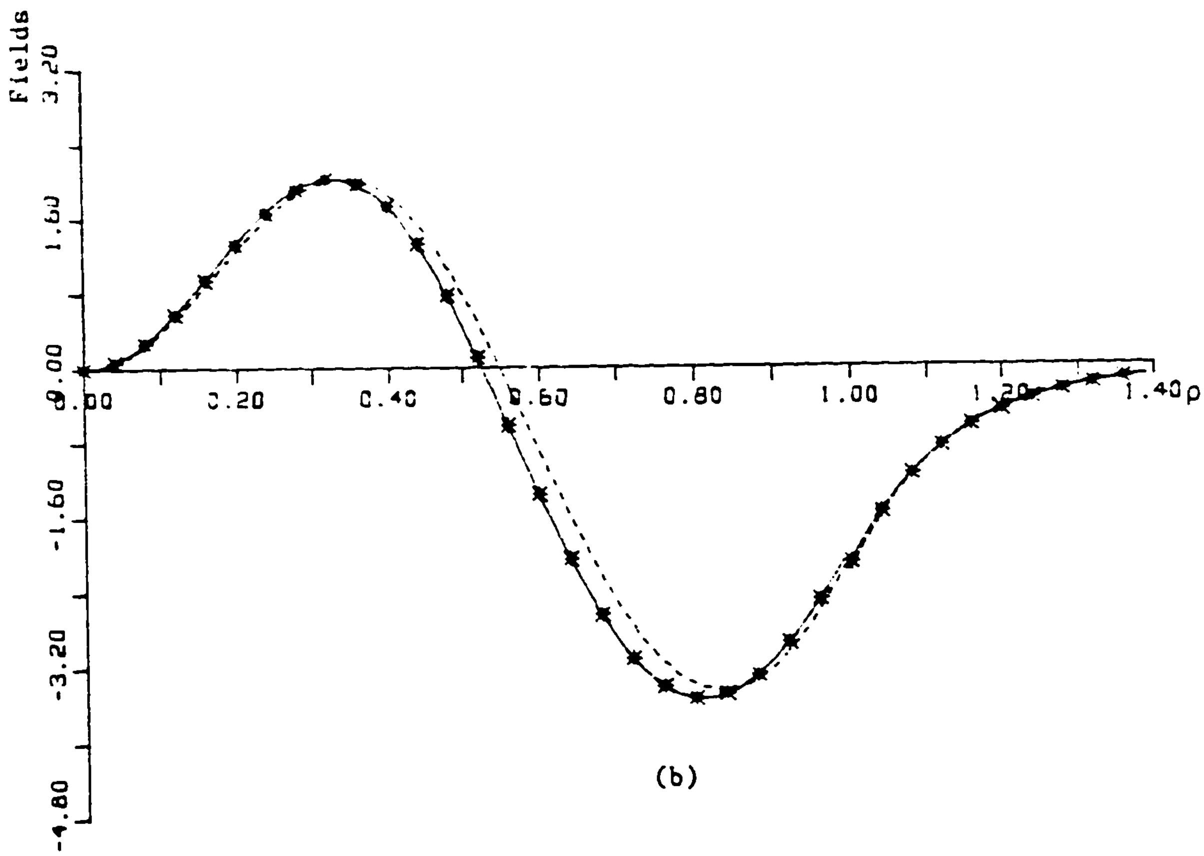
(f)



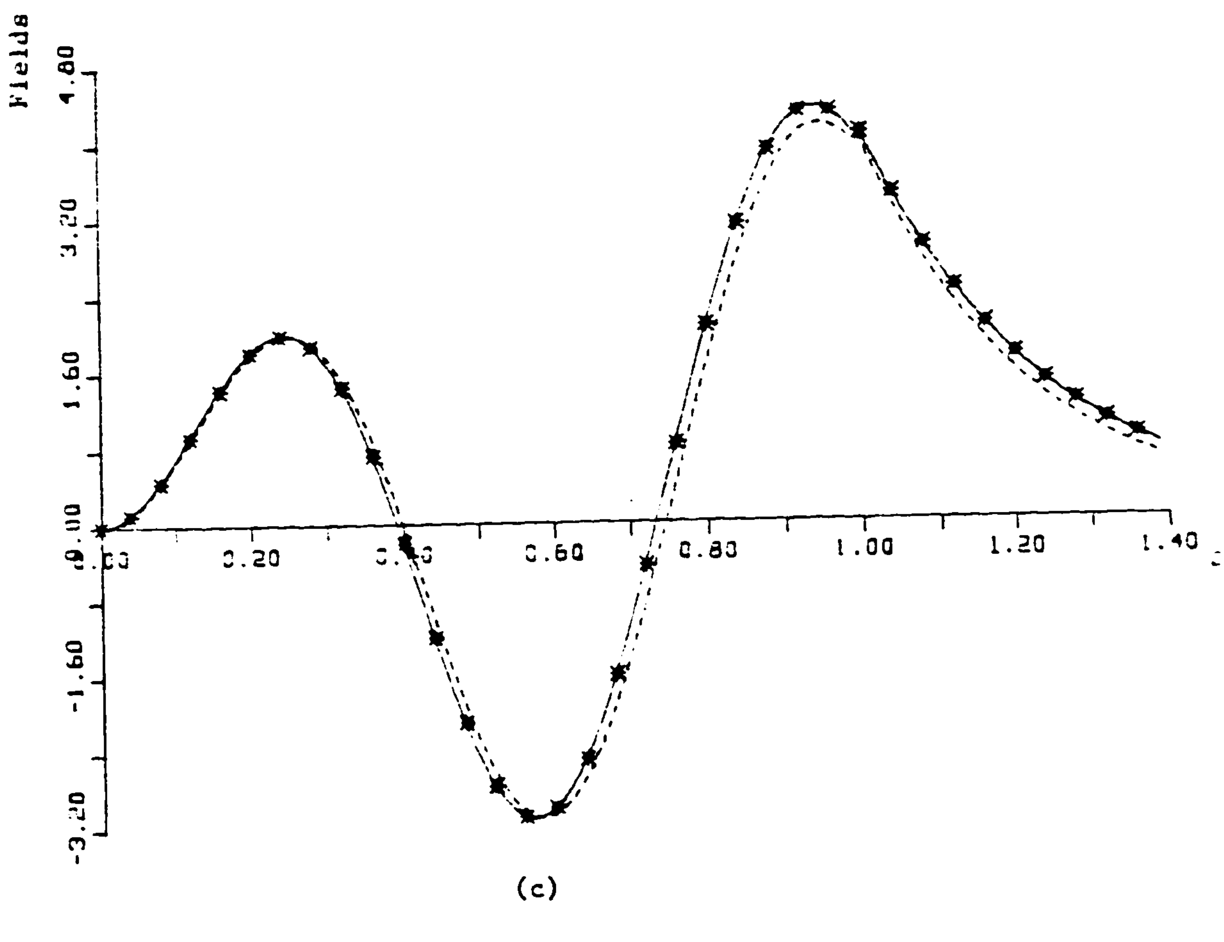
(a)

Figure 11. Field distributions for Quartic Perturbations

First mode for p = 4.



Second mode for p = 4.



(c)

Third mode for p = 4.

unperturbed TE fields. These are shown on the same figure for the purpose of comparison.

An important factor to be considered was the rapidity of convergence of the series in eqn. (4-3-1). It was noted that the convergence strongly depends on 'c' or in other words, on the ratio of the radius 'a' of the fiber to the free space wavelength λ_o. For large values of $\frac{a}{\lambda_o}$, the coefficients in the series become extraordinarily large and an inconvenient number of terms were required for proper accuracy. During different runs of the computer program, it was found that for $\frac{a}{\lambda_o} = 10$, less than 30 terms were required. For $\frac{a}{\lambda_o} < 10$ (usual practical values at optical frequencies) even lesser terms would be sufficient.

The limitation imposed on the rate of convergence of the power series solution makes this method more effective in the evaluation of the less tightly bound modes, and most suitable for the evaluation of surface waves. This feature reinforces the findings of Dil & Blok |40| and Kirchoff |56| regarding the effect of the shape of the refractive index profile on the convergence of the power series expansion.

5.0 CONCLUSIONS AND RECOMMENDATIONS

5.1 CONCLUSIONS

The problem of finding the propagation constants and field distributions for the case of guided electromagnetic waves in inhomogeneous dielectric media with perturbed-parabolic refractive index distribution has been studied in detail. The analysis is presented for an unbounded as well as a cladded medium. A perturbation method has been used for the case of the infinite medium whereas a numerical technique has been applied to the cladded medium. Linear, cubic and quartic perturbations on the parabola have been treated and results are reported. The following conclusions can be drawn from the present study.

(a) For the infinite medium, results of the quartic case compare well with the published results. A detailed discussion of this is given in chapter 3 with the comparison table.

(b) The perturbation calculations using a 6x6 matrix, i.e. with N=6, work well for the lower modes. For higher order modes, the results are relatively less accurate. This effect is more pronounced for eigenvectors than for eigenvalues. To increase the accuracy for higher order modes, the value of N must be increased.

(c) It was noted that only a few unperturbed modes are required to completely describe the perturbed modes for the infinite medium. Hence an infinite series is not necessary because the contributions of the terms that are neglected is extremely small.

(d) For the case of the cladded fiber, the power series method of expansion is particularly suitable for the computation of surface waves. The method is not recommendable for tightly bound modes because the coefficients in the power series become very large and an inconvenient number of terms will be required for the proper convergence. Hence a different method should be sought for calculating such modes.

(e) The shape of the profile and the amount of perturbation also effect the results. For large amount of perturbations the results are not dependable.

5.2 RECOMMENDATIONS FOR FUTURE WORK

Future work may include the following:

(a) The present work can be repeated by considering the y (or ϕ) dependence of fields into account. It must be remembered, however, that in this case, the resulting field distributions will also contain hybrid modes.

(b) A study of material dispersion and its effect on the field distributions for the case of perturbed-parabolic media will be a remarkable addition to the present work.

(c) Some other numerical technique may also be used, particularly to investigate the tightly bound modes.

(d) The media in the present study have been assumed to be lossless. Effect of loss can be considered as an extension.

(e) The present analysis made no approximations for the permittivity profile. A different analysis can be carried out by using a piece-wise linear model for the profile. The analysis using a staircase function approximation for a parabolic profile already exists in the literature |38|.

6.0 APPENDICES

"A-1"

VECTOR IDENTITIES

$$\nabla \times \nabla \times \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} \tag{A-1-1}$$

$$\nabla \times (f\bar{A}) = (\nabla f) \times \bar{A} + f(\nabla \times \bar{A}) \tag{A-1-2}$$

$$\nabla \cdot (f\bar{A}) = (\nabla f) \cdot \bar{A} + f(\nabla \cdot \bar{A}) \tag{A-1-3}$$

"A-2"

COMPUTER PROGRAMS AND RELEVANT SUBROUTINES

Complete listings of all the computer programs used (written in the FORTRAN IV language) and some relevant subroutines is given in the following pages. The working of these programs has been described in the text. Details about the SSP subroutines used can be found in the IBM publication on SSP subroutines. The routines related to the CIL graphical plotter are described in the CIL manuals.

```
C     ------------------------------------------------------------------
C
C     ****  FORTRAN PROGRAM FOR CALCULATING THE NORMALIZED EIGENVALUES
C     ****  ' BR '  AND NORMALIZED EIGENVECTORS ' AR '.
C     ****  THIS PROGRAM ALSO CALCULATES AND PLOTS THE UNPERTURBED
C     ****  FIELDS ' FN ' AND THE PERTURBED FIELDS ' FR '.
C     ****  THE PERTURBATION TERM IS GEMA*(G*X)**LP, WHERE
C     ****  LP=1   MEANS   LINEAR PERTURBATION ON THE PARABOLA,
C     ****  LP=3   MEANS   CUBIC PERTURBATION ON THE PARABOLA, AND
C     ****  LP=4   MEANS   QUARTIC PERTURBATION ON THE PARABOLA.
C     ****  FOR OTHER VALUES OF LP, THE PROGRAM ASSUMES NO PERTURBATION.
C     ****  I AND J ARE COUNTERS THROUGHOUT FOR INDEXING.
C
C     ------------------------------------------------------------------
C
C
C
      REAL A(6,6),A1(6,6),ARN(6,6),B1(6),BR(6),C(6,6),FN(100,6)
     @,FR(100,6),HK(6),KAY,NZERO,PI,PLT(100,3),U(6,6),XSE1(100)
C
C
C        DEFINING STATEMENT FUNCTIONS AND CONSTANTS
C
C
      BEE(NB)=2.E-4*(2*NB+1)
      PI=3.1415927
      NZERO=1.0
      FREQ=1.E12
      CEE=3.E8
      KAY=2.*PI*FREQ/CEE*NZERO
      GEE=2.E-4*KAY
      SZERO=(KAY*GEE)**(-0.5)
C
C
C        SELECTING ORDER OF PERTURBATION
C
C
      LP=4
      GEMA=500
C
C
C
      WRITE(6,1000)GEMA,CEE,FREQ,GEE,KAY,NZERO,PI,SZERO
1000  FORMAT(1H1,/,T20,'THE FOLLOWING CONSTANTS ARE ASSUMED',//////,
     @T30,'GEMA      =',G16.10,///,
     @T30,'C         =',G16.10,///,
     @T30,'FREQUENCY =',G16.10,///,
     @T30,'G         =',G16.10,///,
     @T30,'K         =',G16.10,///,
     @T30,'N-ZERO    =',G16.10,///,
     @T30,'PI        =',G16.10,///,
     @T30,'S-ZERO    =',G16.10,///)
C
C
C         CALCULATING THE MATRIX   A   IN STEPS
C         IT IS A (NMAX X NMAX) MATRIX
C
C
```

```
      NMAX=6
C
C
C        CALCULATING COEFFICIENTS IN THE EXPRESSION OF THE INTEGRAL I(P)
C        IN ORDER TO FORM  THE MATRIX C(M,N)
C        NOTE THAT M=I-1  AND N=J-1 ALWAYS
C        HOWEVER SINCE THERE IS NO C(0,0) FOR THE COMPUTER
C        SO WE HAVE TO USE C(I,J) FOR CALCULATIONS
C
C
      DO 1 I=1,NMAX
      DO 1 J=1,NMAX
1     C(I,J)=0.0
      IF(LP.EQ.4)GO TO 4
      IF(LP.EQ.3)GO TO 5
      IF(LP.EQ.1)GO TO 6
      GO TO 7
C
C
C        CALCULATING THE ELEMENTS OF MATRIX C FOR LP=4
C
C
  4   C(1,1)=GAMMA(2.5)
      C(2,2)=2.*GAMMA(2.5)+12.*GAMMA(1.5)
      C(3,3)=2.*GAMMA(2.5)+24.*GAMMA(1.5)+6.*GAMMA(0.5)
      C(4,4)=4./3.*GAMMA(2.5)+24.*GAMMA(1.5)+12.*GAMMA(0.5)
      C(5,5)=2./3.*GAMMA(2.5)+16.*GAMMA(1.5)+12.*GAMMA(0.5)
      C(6,6)=4./15.*GAMMA(2.5)+8.*GAMMA(1.5)+8.*GAMMA(0.5)
C
C
C
      C(1,3)= 6.*GAMMA(1.5)
      C(3,1)=C(1,3)
      C(1,5)= GAMMA(0.5)
      C(5,1)=C(1,5)
      C(2,4)=12.*GAMMA(1.5)+4.*GAMMA(0.5)
      C(4,2)=C(2,4)
      C(2,6)= 2.*GAMMA(0.5)
      C(6,2)=C(2,6)
      C(3,5)=12.*GAMMA(1.5)+8.*GAMMA(0.5)
      C(5,3)=C(3,5)
      C(4,6)= 8.*GAMMA(1.5)+8.*GAMMA(0.5)
      C(6,4)=C(4,6)
      GO TO 7
C
C
C        CALCULATING THE ELEMENTS OF MATRIX C FOR LP=3
C
C
  5   C(1,2)= 3.*GAMMA(1.5)
      C(2,1)=C(1,2)
      C(1,4)= GAMMA(0.5)
      C(4,1)=C(1,4)
      C(2,3)= 6.*GAMMA(1.5)+3.*GAMMA(0.5)
      C(3,2)=C(2,3)
      C(2,5)= 2.*GAMMA(0.5)
      C(5,2)=C(2,5)
      C(3,4)= 6.*GAMMA(1.5)+6.*GAMMA(0.5)
```

```
      C(4,3)=C(3,4)
      C(3,6)= 2.*GAMMA(0.5)
      C(6,3)=C(3,6)
      C(4,5)= 4.*GAMMA(1.5)+6.*GAMMA(0.5)
      C(5,4)=C(4,5)
      C(5,6)= 2.*GAMMA(1.5)+4.*GAMMA(0.5)
      C(6,5)=C(5,6)
      GO TO 7
C
C
C        CALCULATING THE ELEMENTS OF MATRIX C FOR LP=1
C
C
  6   RTPI=GAMMA(0.5)
      C(1,2)= RTPI
      C(2,1)=C(1,2)
      C(2,3)= 2.*RTPI
      C(3,2)=C(2,3)
      C(3,4)= 2.*RTPI
      C(4,3)=C(3,4)
      C(4,5)= 4./3.*RTPI
      C(5,4)=C(4,5)
      C(5,6)= 2./3.*RTPI
      C(6,5)=C(5,6)
C
C
C
  7   PRINT 1004
1004  FORMAT(1H1,T32,'MATRIX C',//)
      WRITE(6,1001)((C(I,J),J=1,NMAX),I=1,NMAX)
1001  FORMAT(6F12.7 ,//)
C
C
C        CALCULATING THE MATRIX   A   FROM MATRIX  C
C
C
      DO 2 I=1,NMAX
      DO 2 J=1,NMAX
      N=J-1
      IF(I.EQ.J)GO TO 3
      IF(C(I,J).EQ.0.)GO TO 2
      A(I,J)=GEMA*(GEE*SZERO)**LP*GAMMA(FLOAT(I))*C(I,J)/2.**N/
     @PI**0.5
      GO TO 2
3     A(I,J)=BEE(N)+GEMA*GEE**LP*SZERO**LP*GAMMA(FLOAT(I))*C(I,J)/
     @2.**N/PI**0.5
2     CONTINUE
C
C
C        CALCULATING  EIGENVALUES  AND  EIGENVECTORS  OF  MATRIX  A
C
C
      DO 51 I=1,NMAX
      DO 51 J=1,NMAX
      IF(I.EQ.J)GO TO 52
      U(I,J)=0.0
      GO TO 51
52    U(I,J)=1.0
```

```
51    CONTINUE
C
C
C
      DO 53 I=1,NMAX
      DO 53 J=1,NMAX
53    A1(I,J)=A(I,J)
C
      CALL NROOT(NMAX,A1,U,BR,ARN)
C
C
C        REARRANGING BR'S IN AN INCREASING ORDER, AND REORDERING ARN
C        BY TRANSFERING TO DUMMY MATRICES B1 AND A1
C        TO MAINTAIN CONSISTENCY WITH BN'S ( WHICH ARE INCREASING)
C
C
      DO 54 I=1,NMAX
      I54=NMAX+1-I
      B1(I)=BR(I54)
      DO 54 J=1,NMAX
54    A1(J,I)=ARN(J,I54)
C
C
C
      PRINT 1010
1010  FORMAT(1H1,T32,'MATRIX U',//)
      WRITE(6,1001)((U(I,J),J=1,NMAX),I=1,NMAX)
      PRINT 1002
1002  FORMAT(1H1,T32,'MATRIX  A ',//)
      WRITE(6,1001)((A(I,J),J=1,NMAX),I=1,NMAX)
      PRINT 1020
1020  FORMAT(1H0,/,T18,'EIGEN VALUES OF MATRIX A I.E., BR''S',//)
      WRITE(6,1001)(B1(I),I=1,NMAX)
      PRINT 1030
1030  FORMAT(1H0,/,T18,'EIGENVECTORS OF MATRIX A COLUMNWISE.',//)
      WRITE(6,1001)((A1(I,J),J=1,NMAX),I=1,NMAX)
C
C
C        CALCULATING UNPERTURBED FIELDS.  USING THE RELATION
C        FN(XSE)=EXP(-XSE**2/2)*H(K,XSE)*2**(-N/2)
C        NOTE :- THIS IS THE ALTERNATE DEFINITION FOR HERMITE
C
C
      PRINT 1040
1040  FORMAT(1H1,T40,'TABLE OF VALUES FOR UNPERTURBED FIELDS',////)
      PRINT 1041
1041  FORMAT(T4,'XSE',T12,'N = 0',T32,'N = 1',T52,'N = 2',T72,'N = 3'
     a,T92,'N = 4',T112,'N = 5',//)
C
C
      DO 101 J=1,100
      XSE1(J)=FLOAT(J-50)/10.
      XSE=XSE1(J)
C
      CALL HEP(HK,XSE,5)
C
      DO 101 I=1,NMAX
101   FN(J,I)=HK(I)*EXP(-XSE**2/2.)*2.**(-FLOAT(I-1)/2.)
```

```
C
C
      DO 102 J=1,100
102   WRITE(6,1005)XSE1(J),(FN(J,I),I=1,NMAX)
1005  FORMAT(F6.2,6G20.12,//)
C
C
C        CALCULATING THE PERTURBED FIELDS
C        FROM EIGENVECTORS AND UNPERTURBED FIELDS
C        USING THE RELATION < FR > = < FN >< ARN >
C
C
      CALL GMPRD(FN,A1,FR,100,NMAX,NMAX)
C
C
C
      PRINT 1050
1050  FORMAT(1H1,T40,'TABLE OF VALUES FOR PERTURBED FIELDS',/////)
      PRINT 1041
      DO 103 J=1,100
103   WRITE(6,1005)XSE1(J),(FR(J,I),I=1,NMAX)
C
C
C        PLOTTING THE UNPERTURBED AND PERTURBED FIELDS
C
C
      DO 104 I=1,NMAX
      DO 105 J=1,100
      PLT(J,1)=XSE1(J)
      PLT(J,2)=FN(J,I)
105   PLT(J,3)=FR(J,I)
C
      CALL PLOT(I-1,PLT,100,3,100,0)
104   CONTINUE
C
C
C
      STOP
      END
```

```
C-----------------------------------------------------------------------
C
C                         SUBROUTINE PLOT
C
C-----------------------------------------------------------------------
C
C     PURPOSE:
C     PLOT SEVERAL CROSS-VARIABLES VERSUS A BASE VARIABLE
C
C     USAGE:
C     CALL PLOT (NO,A,N,M,NL,NS)
C
C     DESCRIPTION OF VARIOUS PARAMETERS:
C
C     NO - CHART NUMBER (3 DIGITS MAXIMUM)
C
C     A  - MATRIX OF DATA TO BE PLOTTED. FIRST COLUMN REPRESENTS
C          BASE VARIABLE AND SUCCESSIVE COLUMNS ARE THE CROSS-
C          VARIABLES (MAXIMUM IS 9)
C
C     N -  NUMBER OF ROWS IN MATRIX A (THERE IS NO LIMIT ON THIS)
C
C     M -  NUMBER OF COLUMNS IN MATRIX A (EQUAL TO THE TOTAL
C          NUMBER OF VARIABLES). MAXIMUM IS 10
C
C     NL - NUMBER OF LINES IN THE PLOT. IF 0 IS SPECIFIED, 50
C          LINES ARE USED( USUALLY TAKE NL = N ---- BUT IN NO CASE
C          N > NL; HOWEVER   N <= NL  CAN WORK. FOR NL=0, THE GRAPH
C          IS CONFINED TO A SINGLE PAGE)
C
C     NS - CODE FOR SORTING THE BASE VARIABLE DATA IN ASCENDING
C          ORDER
C
C            0   SORTING IS NOT NECESSARY (ALREADY IN ASCENDING
C                ORDER (0 OR ANY NEGATIVE INTEGER)
C            1   SORTING IS NECESSARY ( 1 OR ANY POSITIVE INTEGER)
C
C-----------------------------------------------------------------------
C
C
      SUBROUTINE PLOT(NO,A,N,M,NL,NS)
      DIMENSION OUT(101),YPR(11),ANG(9),A(1)
C
C
C        THE CHARACTERS FOR THE GRAPHS CAN BE CHANGED ALSO ------IN THE
C        FOLLOWING DATA STATEMENT.
C
C
      DATA BLANK/' '/,ANG/'*','+','.','0','a','#','?','6','|'/
    1 FORMAT(1H1,40X,'GRAPH FOR R  = ',I3,T80,'FIELDS-------->',//)
    2 FORMAT(1H ,F11.4,5X,101A1)
    3 FORMAT(1H )
    7 FORMAT(1H ,16X,101H+         +         +         +         +
     a    +         +         +         +         +         +)
    8 FORMAT(1H0,9X,11F10.4)
      NLL=NL
C
C
```

```
      IF(NS) 16, 16, 10
C
C        SORT BASE VARIABLE IN ASCENDING ORDER
C
   10 DO 15 I=1,N
      DO 14 J=I,N
      IF(A(I)-A(J)) 14, 14, 11
   11 L=I-N
      LL=J-N
      DO 12 K=1,M
      L=L+N
      LL=LL+N
      F=A(L)
      A(L)=A(LL)
   12 A(LL)=F
   14 CONTINUE
   15 CONTINUE
C
C           TEST NLL
C
   16 IF(NLL) 20, 18, 20
   16 NLL=50
C
C           PRINT TITLE
C
   20 WRITE(6,1)NO
C
C           FIND SCALE FOR BASE VARIABLE
C
      XSCAL=(A(N)-A(1))/(FLOAT(NLL-1))
C
C           FIND SCALE FOR CROSS-VARIABLES
C
      M1=N+1
      YMIN=A(M1)
      YMAX=YMIN
      M2=M*N
      DO 40 J=M1,M2
      IF(A(J)-YMIN) 28,26,26
   26 IF(A(J)-YMAX) 40,40,30
   26 YMIN=A(J)
      GO TO 40
   30 YMAX=A(J)
   40 CONTINUE
      YSCAL=(YMAX-YMIN)/100.00
C
C        PRINT CROSS-VARIABLE NUMBERS
C
      YPR(1)=YMIN
      DO 90 KN=1,9
   90 YPR(KN+1)=YPR(KN)+YSCAL*10.0
      YPR(11)=YMAX
      WRITE(6,8)(YPR(IP),IP=1,11)
      WRITE(6,7)
C
C        FIND BASE VARIABLE PRINT POSITION
C
      XB=A(1)
```

```
      L=1
      MY=M-1
      I=1
   45 F=I-1
      XPR=XB+F*XSCAL
      IF(A(L)-XPR) 50,50,70
C
C        FIND CROSS-VARIABLES
C
   50 DO 55 IX=1,101
   55 OUT(IX)=BLANK
      DO 60 J=1,MY
      LL=L+J*N
      JP=((A(LL)-YMIN)/YSCAL)+1.0
      OUT(JP)=ANG(J)
   60 CONTINUE
C
C        FINDIMG AXIS FOR BASE VARIABLE.
C
      IF(YMIN.GT.0.)GO TO 100
      KP=1.-YMIN/YSCAL
      OUT(KP)=ANG(9)
C
C        PRINT LINE AND CLEAR, OR SKIP
C
  100 WRITE(6,2)XPR,(OUT(IZ),IZ=1,101)
      L=L+1
      GO TO 80
   70 WRITE(6,3)
   80 I=I+1
      IF(I-NLL) 45, 84, 86
   84 XPR=A(N)
      GO TO 50
C
C        PRINT CROSS-VARIAELE NUMBERS AGAIN
C
   86 WRITE(6,7)
      WRITE(6,8)(YPR(IP),IP=1,11)
C
C
C
      RETURN
      END
```

```
C     ----------------------------------------------------------------
C
C     ****  FORTRAN PROGRAM FOR PLOTTING A GRAPH OF
C     ****  THE CHARACTERISTIC EQUATION VERSUS ALPHA
C     ****  TO EVALUATE THE NORMALIZED EIGENVALUES(I.E. ALPHA)
C     ****  FOR THE CASE OF A GRADED-CORE OPTICAL FIBRE
C     ****  HAVING A PARABOLIC REFRACTIVE INDEX
C     ****  WITH POLYNOMIAL TYPE PERTURBATIONS ON IT
C     ****  ALPHA VARIES IN STEPS OF DEL , WHERE
C     ****  APPROXIMATE VALUE OF DEL=(ALPHAU-ALPHAL)/LMAX
C     ****  MINIMUM VALUE OF ALPHA IS N-CLADDING
C     ****  MAXIMUM VALUE OF ALPHA IS N-CORE(I.E. N-ZERO)
C     ****  M CAN BE ZERO OR ANY +VE INTEGER.
C
C     ----------------------------------------------------------------
C
C
C
      REAL G(100,3),NCSQ,NZSQ,T1(4,4),T2(4,4)
      DIMENSION A(31,4,4),A0(4,4),AI(4,4),AL(31,4),BL(31,4),
     a          TL1(4),TL2(4),TL3(4),TL4(4),TL5(4),TL6(4),TL7(4),TL8(4)
C
C
C
      IIMAX=31
      LMAX=100
C
C
C        DEFINING CONSTANTS AND TYPE OF PERTURBATION
C
C
      ALPHAL=1.50
      ALPHAU=1.51
      C=6.283*10.0
      D1=0.0
      D2=0.003
      D3=0.001
      D4=0.0
      DEL=(ALPHAU-ALPHAL)/LMAX
      NCSQ=ALPHAL**2
      NZSQ=ALPHAU**2
      WRITE(6,*)C,D1,D2,D3,D4,DEL,NCSQ,NZSQ
C
C
C
      M=0
      ALPHA=ALPHAL
C
C
C
      DO 2 L=1,LMAX
      ALPHA=ALPHA+DEL
      BZ2=NZSQ-ALPHA**2
      IF(BZ2)15,16,16
   15 BZ=(-BZ2)**0.5
      GO TO 17
   16 BZ=BZ2**0.5
   17 BONEZ=(ALPHA**2-NCSQ)*C**2
```

```
      BONE=BONE2**0.5
C
C
C        CALCULATING GEMA-SUB-M
C
C
      X=BONE
      CALL BESK(X,M,BK,IER)
      MP=M+1
      CALL BESK(X,MP,BKP,IER)
      IF (M.EQ.0) GO TO 20
C
C        BECAUSE K(+1)=K(-1)
C
      MN=M-1
      CALL BESK(X,MN,BKN,IER)
      GEMA=-X/2.*(BKP+BKN)/BK
      GO TO 10
   20 GEMA=-X*BKP/BK
C
C
C        CALCULATING COMPONENTS OF COLUMN VECTORS 'L'
C
C
   10 A(1,1,2)=-C*BZ2/NZSQ
      A(1,1,4)=-M*ALPHA/NZSQ
      A(1,2,1)=-M**2/C
      A(1,2,3)=M*ALPHA
      A(1,3,2)=M*ALPHA/NZSQ
      A(1,3,4)=-M**2/C/NZSQ
      A(1,4,1)=-M*ALPHA
      A(1,4,3)=-C*BZ2
C
C
C
      A(2,1,2)=C*D1*ALPHA**2/NZSQ
      A(2,1,4)=-M*D1*ALPHA/NZSQ
      A(2,2,1)=0.0
      A(2,2,3)=0.0
      A(2,3,2)=M*D1*ALPHA/NZSQ
      A(2,3,4)=-M**2*D1/C/NZSQ
      A(2,4,1)=0.0
      A(2,4,3)=C*D1*NZSQ
C
C
C
      A(3,1,2)=C*D2*ALPHA**2/NZSQ
      A(3,1,4)=-M*D2*ALPHA/NZSQ
      A(3,2,1)=C*NZSQ
      A(3,2,3)=0.0
      A(3,3,2)=M*D2*ALPHA/NZSQ
      A(3,3,4)=C-M**2*D2/C/NZSQ
      A(3,4,1)=0.0
      A(3,4,3)=C*D2*NZSQ
C
C
C
      A(4,1,2)=C*D3*ALPHA**2/NZSQ
```

```
      A(4,1,4)=-M*D3*ALPHA/NZSQ
      A(4,2,1)=-C*D1*NZSQ
      A(4,2,3)=0.0
      A(4,3,2)=M*D3*ALPHA/NZSQ
      A(4,3,4)=-M**2*D3/C/NZSQ
      A(4,4,1)=0.0
      A(4,4,3)=C*D3*NZSQ
C
C
C
      A(5,1,2)=C*D4*ALPHA**2/NZSQ
      A(5,1,4)=-M*D4*ALPHA/NZSQ
      A(5,2,1)=-C*D2*NZSQ
      A(5,2,3)=0.0
      A(5,3,2)=M*D4*ALPHA/NZSQ
      A(5,3,4)=-M**2*D4/C/NZSQ
      A(5,4,1)=0.0
      A(5,4,3)=C*D4*NZSQ
C
C
C
      A(6,2,1)=-C*D3*NZSQ
      A(7,2,1)=-C*D4*NZSQ
C
C
C         READING VALUES OF L AT RHO=0.0
C
C
      AL(1,1)=-BZ2
      AL(1,2)=M*NZSQ/C
      AL(1,3)=M*ALPHA/C
      AL(1,4)=0.0
      BL(1,1)=0.0
      BL(1,2)=M*ALPHA/C
      BL(1,3)=M/C
      BL(1,4)=-BZ2
C
C
C
      DO 5I=1,4
      DO 5J=1,4
    5 AO(I,J)=A(1,I,J)
C
C
C
      DO 40 II=2,IIMAX
C
      DO 6 I=1,4
      DO 6 J=1,4
      IF(I.NE.J)GO TO 41
      AI(I, )=1.0*(M+II-1)
      GO TO 6
   41 AI(I,J)=0.0
    6 CONTINUE
C
      CALL GMADD(AO,AI,T1,4,4)
      Y=(II-1)*(2*M+II-1)
      CALL SDIV(T1,Y,AI,4,4,0)
```

```
C
C
C
      DO 7 J=1,4
      TL5(J)=0.0
    7 TL6(J)=0.0
C
C
C
      DO 50JJ=2,II
C
      DO 8 I=1,4
      DO 8 J=1,4
    8 T2(I,J)=A(JJ,I,J)
C
C
C
      DO 9J=1,4
      TL1(J)=AL(II-JJ+1,J)
    9 TL2(J)=BL(II-JJ+1,J)
      CALL GMPRD(T2,TL1,TL3,4,4,1)
      CALL GMPRD(T2,TL2,TL4,4,4,1)
C
C
C
      DO 11 J=1,4
      TL5(J)=TL3(J)+TL5(J)
   11 TL6(J)=TL4(J)+TL6(J)
C
   50 CONTINUE
C
C
C
      CALL GMPRD(AI,TL5,TL7,4,4,1)
      CALL GMPRD(AI,TL6,TL8,4,4,1)
C
      DO 12 J=1,4
      AL(II,J)=TL7(J)
   12 BL(II,J)=TL8(J)
C
   40 CONTINUE
C
C
C        ADDING THE SERIES AT RHO=1.0
C
C
      DO 13 I=2,IIMAX
      Y=1.0
      CALL SRMA(AL,Y,IIMAX,4,I,1)
   13 CALL SRMA(BL,Y,IIMAX,4,I,1)
C
C
C
      IF(L.EQ.1)GO TO 200
      IF(L.EQ.LMAX/3)GO TO 200
      IF(L.EQ.2*LMAX/3)GO TO 200
      IF(L.EQ.LMAX)GO TO 200
      GO TO 203
```

```
 200   WRITE(6,201)((AL(I,J),J=1,4),I=1,IIMAX)
 201   FORMAT('  ---------------->TM MODES,I.E.,AL',10X,4G15.7)
       WRITE(6,202)((BL(I,J),J=1,4),I=1,IIMAX)
 202   FORMAT('  TE MODES,I.E.,BL',10X,4G15.7)
C
C
C
 203   RLA1=AL(1,1)
       RLA2=AL(1,2)
       RLA3=AL(1,3)
       RLA4=AL(1,4)
       RLB1=BL(1,1)
       RLB2=BL(1,2)
       RLB3=BL(1,3)
       RLB4=BL(1,4)
C
C
C         EVALUATING THE CHARACTERISTIC EQUATION FOR PLOTTING
C
C
 21    G(L,1)=ALPHA
       IF(M.NE.0)GO TO 30
C
C
C         SEPERATING TE AND TM MODES FOR M=0
C
C
       G(L,2)=RLA1*NCSQ*GEMA-BONE2*RLA2/C
       G(L,3)=RLB4*GEMA-BONE2*RLB3/C
       GO TO 33
C
C
C         WHEN M=0  G(L,2) CONTAINS TM AND G(L,3) CONTAINS TE , BUT
C         WHEN M>0  G(L,2) CONTAINS 0  AND G(L,3) CONTAINS THE
C         DIFFERENCE OF L.H.S AND R.H.S OF THE CHARACTERISTIC EQUATION
C
C
 30    G(L,2)=0.0
       G(L,3)=(BONE2/C*RLA2-M*ALPHA*RLA4-NCSQ*GEMA*RLA1)*
     a   (BONE2/C*RLB3-M*ALPHA*RLB1-GEMA*RLB4)-
     a       (BONE2/C*RLB2-M*ALPHA*RLB4-NCSQ*GEMA*RLB1)*
     a   (BONE2/C*RLA3-M*ALPHA*RLA1-GEMA*RLA4)
C
C
C
 33    WRITE(6,101)M,G(L,1),BONE2,X,BKN,BK,BKP
       WRITE(6,102)RLA1,RLA2,RLA3,RLA4,GEMA
       WRITE(6,103)RLB1,RLB2,RLB3,RLB4,G(L,2),G(L,3)
101    FORMAT(1H0,I3,6G15.6)
102    FORMAT(1H ,T35,5G15.6)
103    FORMAT(1H ,T40,6G15.6)
C      WRITE(15,155)(G(L,LP),LP=1,3)
155    FORMAT(3G14.7)
2      CONTINUE
       CALL PLOT(M,G,LMAX,3,LMAX,0)
       STOP
       END
```

```
C     ----------------------------------------------------------------
C
C     ****  FORTRAN PROGRAM FOR PLOTTING THE FIELDS 'L'
C     ****  AGAINST THE NORMALIZED RADIUS 'RHO'
C     ****  USING THE SERIES EXPANSION OF 'L' IN TERMS OF 'RHO'
C     ****  FOR THE CASE OF A GRADED-CORE OPTICAL FIBRE
C     ****  HAVING A PARABOLIC REFRACTIVE INDEX
C     ****  WITH POLYNOMIAL TYPE PERTURBATIONS ON IT
C     ****  RHO VARIES IN STEPS OF DRHO=0.01
C     ****  MINIMUM VALUE OF ALPHA IS N-CLADDING
C     ****  MAXIMUM VALUE OF ALPHA IS N-CORE(I.E. N-ZERO)
C     ****  M CAN BE ZERO OR ANY +VE INTEGER
C
C     ----------------------------------------------------------------
C
C
C
      REAL NCSQ,NZSQ,T1(4,4),T2(4,4)
      DIMENSION  ALFA(4),ALFAP(3,4),D(3),PLT(100,3)
      DIMENSION A(31,4,4),AO(4,4),AI(4,4),AL(31,4),BL(31,4),
     a          TL1(4),TL2(4),TL3(4),TL4(4),TL5(4),TL6(4),TL7(4),TL8(4)
      DATA ALFA/1.50616,1.50569,1.50216,1.50/,D/0.0003,0.001,0.002/
C
C
C
      IIMAX=31
      NPTS=100
      NMODE=4
C
C
C
      ALFAP(1,1)=1.50804
      ALFAP(1,2)=1.50556
      ALFAP(1,3)=1.50207
      ALFAP(1,4)=1.50000
      ALFAP(2,1)=1.50802
      ALFAP(2,2)=1.50546
      ALFAP(2,3)=1.50180
      ALFAP(2,4)=1.50000
      ALFAP(3,1)=1.50797
      ALFAP(3,2)=1.50532
      ALFAP(3,3)=1.50183
      ALFAP(3,4)=1.50000
C
C
C        ASSUMING CONSTANTS
C
C
      ALPHAL=1.50
      ALPHAU=1.51
      C=6.283*10.0
      D2=0.003
      DRHO=0.01
      NCSQ=ALPHAL**2
      NZSQ=ALPHAU**2
C
C
C
```

```
      M=0
C
C
C
      DC 300 IPERT=1,4
      JPERT=IPERT-1
      DO 300 IMODE=1,NMODE
      ALPHA=ALFA(IMODE)
      DO 1 ICRVS=1,2
      WRITE(6,*)ALPHA,C,D2,NCSQ,NZSQ
C
C
C
      BZ2=NZSC-ALPHA**2   .
      BONE2=(ALPHA**2-NCSQ)*C**2
      BONE=BONE2**0.5  .
C
C
C         CALCULATING GEMA-SUB-M
C
C
      X=BONE
      CALL BESK(X,M,BK,IER)
      MP=M+1
      CALL BESK(X,MP,BKP,IER)
      IF (M.EQ.0) GO TO 20
C
C         BECAUSE K(+1)=K(-1)
C
      MN=M-1
      CALL BESK(X,MN,BKN,IER)
      GEMA=-X/2.*(BKP+BKN)/BK
      GO TO 10
   20 GEMA=-X*BKP/BK
      WRITE(6,101)M,BONE2,X,BKN,BK,BKP,GEMA
  101 FORMAT(1H0,I3,6G15.6)
C
C
C         CALCULATING COMPONENTS OF COLUMN VECTORS ' L '
C
C
   10 A(1,1,2)=-C*BZ2/NZSQ
      A(1,1,4)=-M*ALPHA/NZSC
      A(1,2,1)=-M**2/C
      A(1,2,3)=M*ALPHA
      A(1,3,2)=M*ALPHA/NZSC
      A(1,3,4)=-M**2/C/NZSC
      A(1,4,1)=-M*ALPHA
      A(1,4,3)=-C*BZ2
C
C
C
      A(2,1,2)=C*D1*ALPHA**2/NZSC
      A(2,1,4)=-M*D1*ALPHA/NZSC
      A(2,2,1)=0.0
      A(2,2,3)=0.0
      A(2,3,2)=M*D1*ALPHA/NZSC
      A(2,3,4)=-M**2*D1/C/NZSC
```

```
      A(2,4,1)=0.0
      A(2,4,3)=C*D1*NZSQ
C
C
C
      A(3,1,2)=C*D2*ALPHA**2/NZSQ
      A(3,1,4)=-M*D2*ALPHA/NZSQ
      A(3,2,1)=C*NZSQ
      A(3,2,3)=0.0
      A(3,3,2)=M*D2*ALPHA/NZSQ
      A(3,3,4)=C-M**2*D2/C/NZSQ
      A(3,4,1)=0.0
      A(3,4,3)=C*D2*NZSQ
C
C
C
      A(4,1,2)=C*D3*ALPHA**2/NZSQ
      A(4,1,4)=-M*D3*ALPHA/NZSQ
      A(4,2,1)=-C*D1*NZSQ
      A(4,2,3)=0.0
      A(4,3,2)=M*D3*ALPHA/NZSQ
      A(4,3,4)=-M**2*D3/C/NZSQ
      A(4,4,1)=0.0
      A(4,4,3)=C*D3*NZSQ
C
C
C
      A(5,1,2)=C*D4*ALPHA**2/NZSQ
      A(5,1,4)=-M*D4*ALPHA/NZSQ
      A(5,2,1)=-C*D2*NZSQ
      A(5,2,3)=0.0
      A(5,3,2)=M*D4*ALPHA/NZSQ
      A(5,3,4)=-M**2*D4/C/NZSQ
      A(5,4,1)=0.0
      A(5,4,3)=C*D4*NZSQ
C
C
C
      A(6,2,1)=-C*D3*NZSQ
      A(7,2,1)=-C*D4*NZSQ
C
C
C         READING VALUES OF L AT RHO=0.0
C
C
      AL(1,1)=-BZ2
      AL(1,2)=M*NZSQ/C
      AL(1,3)=M*ALPHA/C
      AL(1,4)=0.0
      BL(1,1)=0.0
      BL(1,2)=M*ALPHA/C
      BL(1,3)=M/C
      BL(1,4)=-BZ2
C
C
C
      DO 5I=1,4
      DO 5J=1,4
```

```
    5 AO(I,J)=A(1,I,J)
C
C
C
      DO 40 II=2,IIMAX
C
      DO 6 I=1,4
      DO 6 J=1,4
      IF(I.NE.J)GO TO 41
      AI(I,J)=1.0*(M+II-1)
      GO TO 6
   41 AI(I,J)=0.0
    6 CONTINUE
C
      CALL GMADD(AO,AI,T1,4,4)
C
      Y=(II-1)*(2*M+II-1)
      CALL SDIV(T1,Y,AI,4,4,0)
C
      DO 7 J=1,4
      TL5(J)=0.0
    7 TL6(J)=0.0
C
C
C
      DO 50JJ=2,II
C
      DO 8 I=1,4
      DO 8 J=1,4
    8 T2(I,J)=A(JJ,I,J)
C
      DO 9J=1,4
      TL1(J)=AL(II-JJ+1,J)
    9 TL2(J)=BL(II-JJ+1,J)
C
      CALL GMPRD(T2,TL1,TL3,4,4,1)
      CALL GMPRD(T2,TL2,TL4,4,4,1)
C
      DO 11 J=1,4
      TL5(J)=TL3(J)+TL5(J)
   11 TL6(J)=TL4(J)+TL6(J)
C
   50 CONTINUE
C
C
C
      CALL GMPRD(AI,TL5,TL7,4,4,1)
      CALL GMPRD(AI,TL6,TL8,4,4,1)
C
      DO 12 J=1,4
      AL(II,J)=TL7(J)
   12 BL(II,J)=TL8(J)
C
   40 CONTINUE
C
C
C          WRITING THE COEFFICIENTS IN THE SERIES EXPANSION AT RHO=1.0
C
```

```
C
      IF(ICRVS.NE.1)GO TO 32
      IF(IPERT.NE.1)GO TO 33
   32 WRITE(6,201)ALPHA,D1,D2,D3,D4
      DO 30 I=1,IIMAX
      INDEX=I-1
   30 WRITE(6,203)INDEX,(AL(I,J),J=1,4)
      WRITE(6,202)ALPHA,D1,D2,D3,D4
      DO 31 I=1,IIMAX
      INDEX=I-1
   31 WRITE(6,203)INDEX,(BL(I,J),J=1,4)
  201 FORMAT(1H1,5X,'COMPONENTS OF COLUMN VECTOR   L   FOR TM MODES WITH
     @ ALPHA =',F8.5,//,16X,'D1 =',F7.4,' D2 =',F7.4,' D3 =',F7.4,
     @' D4 =',F7.4,///)  .
  202 FORMAT(1H1,5X,'COMPONENTS OF COLUMN VECTOR   L   FOR TE MODES WITH
     @ ALPHA =',F8.5,//,16X,'D1 =',F7.4,' D2 =',F7.4,' D3 =',F7.4,
     @' D4 =',F7.4,///)
  203 FORMAT(1H ,5X,'L(',I2,')  ---->  ',4F14.7)
C
C
C         CALCULATING FIELDS AT DIFFERENT RHO'S USING THE SERIES
C
C
   33 DO 14 J=1,4
      TL7(J)=AL(1,J)
   14 TL8(J)=BL(1,J)
C
C
C
      RHO=0.0
      DO 2 L=1,NPTS
C
      DO 13 I=2,IIMAX
      Y=RHO**(I-1)
      CALL SRMA(AL,Y,IIMAX,4,I,1)
   13 CALL SRMA(BL,Y,IIMAX,4,I,1)
C
C
C         ARRANGING THE POINTS IN A MATRIX, FOR PLOTTING THE
C         PHI-COMPONENTS OF THE VECTOR ' L '
C
C
      IF(ICRVS.NE.1)GO TO 15
      PLT(L,1)=RHO
      PLT(L,2)=BL(1,3)
      PLT(L,3)=AL(1,2)
      GO TO 17
C
C
C         FOR THE UNPERTURBED CASE,
C              PLT(L,2) STORES E(PHI) FOR TE MODES,AND
C              PLT(L,3) STORES H(PHI) FOR TM MODES,BUT
C         FOR THE PERTURBED CASE,
C              PLT(L,2) STORES E(PHI) FOR TE (UNPERTURBED)
C              PLT(L,3) STORES E(PHI) FOR TE (PERTURBED)
C         THIS CONFORMS WITH THE RECTANGULAR CASE, WHERE WE PLOT
C         E(Y) FOR TE AND H(Y) FOR TM
C
```

```
C
   15 PLT(L,3)=BL(1,3)
   17 DO 16 J=1,4
      AL(1,J)=TL7(J)
   16 BL(1,J)=TL8(J)
    2 RHO=RHO+DRHO
C
C
C          INJECTING PERTURBATION WHEN IPERT > 1 AND ICRVS > 1
C          QUIT LOOP WHEN NO PERTURBATION (I.E., IPERT = 1)
C
C
      IF(IPERT.EQ.1)GO TO 18
      ALPHA=ALFAP(JPERT,IMODE)
      IF(IPERT.EQ.2)D1=D(JPERT)
      IF(IPERT.EQ.3)D3=D(JPERT)
      IF(IPERT.EQ.4)D4=D(JPERT)
    1 CONTINUE
C
C
C          STORING DATA FOR C.I.L  PLOTTER
C
C
   18 DO 19 I=1,NPTS
   19 WRITE(15,155)(PLT(I,J),J=1,3)
  155 FORMAT(3G14.7)
C
C          RESETTING COEFFICIENTS FOR ANOTHER TYPE OF PERTURBATION
C
C
      D1=0.0
      D3=0.0
      D4=0.0
C
C
C
      CALL PLOT(IMODE,PLT,NPTS,3,NPTS,0)
  300 CONTINUE
      STOP
      END
```

```
C-----------------------------------------------------------------------
C
C                         SUBROUTINE ANJUM
C
C-----------------------------------------------------------------------
C
C     *****        JUMLA HCCUC EAHAC-E-MUSSANIF MAHFOOZ HAIN       *****
C
C     DESCRIPTION:
C
C        THIS IS A GENERAL SUBROUTINE SIMILAR TO 'PLOT' OF 'SSP' USING
C     C.I.L DRIVER LIBRARY ROUTINES.
C
C     THE SUBROUTINE CAN BE USED WITH THE 'SPECIAL PROGRAM' OR CAN BE
C     CALLED LIKE PLOT OF SSP
C
C     IT DRAWS (NCOLS-1) CURVES ON A SINGLE PAIR OF AXES ON EACH CALL
C
C     THE AXES WILL START AT XMIN AND YMIN AND INTERSECT AT (0,0) WHEN
C     XMIN AND YMIN IS NEGATIVE. IN CASE ANY OF THEM IS POSITIVE, THE
C     PARTICULAR AXIS WILL START AT THE EXTREME BOTTOM OR EXTREME LEFT
C
C     THE DIMENSION OF THE ARRAYS X AND Y MUST BE > OR = NPTS
C
C     THE VALUES OF THE BASE VARIABLE MUST BE IN ASCENDING ORDER
C
C
C     ICALL= COUNTER SHOWING THE I(TH) CALL OF THE SUBROUTINE. AT THE
C            BEGINING OF A PROGRAM, ICALL=0. EACH TIME A CALL TO 'ANJUM'
C            IS MADE, ICALL SHOULD BE INCREMENTED BY 1 EXTERNALLY.
C            FOR A NEW PLOTTING PAGE, MAKE ICALL=3
C
C     P    = MATRIX OF DATA TO BE PLOTTED. ITS FIRST COLUMN REPRESENTS
C            BASEVARIABLE AND SUCCESSIVE COLUMNS ARE THE CROSSVARIABLES
C
C     NPTS = NUMBER OF POINTS FOR THE BASE VARIABLE I.E.,ROWS IN P
C
C     NCOLS= NUMBER OF COLUMNS IN P. THUS NUMBER OF CROSS VARIABLES
C            IS (NCOLS-1). NCOLS MUST BE >= 2
C
C     NCHN = AN INTEGER SPECIFYING THAT THE NCHN(TH) COLUMN OF MATRIX P
C            IS TO BE PLOTTED AS A CHAINED CURVE.  2 <= NCHN <= NCOLS
C            IF NCHN IS OUTSIDE THIS RANGE NO CHAINED CURVE IS DRAWN
C
C     XLNTH= LENGTH OF X-AXIS IN CENTIMETERS (WHOLE NUMBER)
C     YLNTH= LENGTH OF Y-AXIS IN CENTIMETERS (WHOLE NUMBER)
C
C     ISCAL= POSITIVE MEANS SCALING IS DONE BY 'SCALE' OF C.I.L.  IN
C            THIS CASE XMIN,XMAX,YMIN,YMAX ARE NOT REQUIRED BUT SOME
C            VALUE MUST BE GIVEN TO THEM
C            ZERO OR NEGATIVE MEANS SCALING IS NOT DONE BY SCALE OF
C            C.I.L. HENCE XMIN,XMAX,YMIN,YMAX ARE REQUIRED
C
C     XMAX = MAXIMUM VALUE OF THE X-AXIS IN USER UNITS
C     YMAX = MAXIMUM VALUE OF THE Y-AXIS IN USER UNITS
C     XMIN = MINIMUM VALUE OF THE X-AXIS IN USER UNITS
C     YMIN = MINIMUM VALUE OF THE   AXIS IN USER UNITS
C
```

```
C-----------------------------------------------------------------------
C
C
C
      SUBROUTINE ANJUM (ICALL,P,NPTS,NCOLS,NCHN,XLNTH,YLNTH,ISCAL,
     a                                           XMIN,XMAX,YMIN,YMAX)
      DIMENSION P(NPTS,NCOLS),X(100 ),Y(100 )
      INTEGER BLACK,BLUE,RED
      BLACK=1
      BLUE=2
      RED=3
C
C        GIVING A SPACE OF 5 CM IN THE Y-DIRECTION BETWEEN NEW GRAPH AND
C        PREVIOUS GRAPH
C
      CALL PLOT(0.0,5.0,-3)
      DO 1 I=1,NPTS
    1 X(I)=P(I,1)
      IF(ISCAL.GT.0)GO TO 10
C
C        FINDING INCREMENTS ON X AND Y AXES PER CENTIMETER
C
      DX=(XMAX-XMIN)/XLNTH
      DY=(YMAX-YMIN)/YLNTH
      GO TO 11
C
C        FINDING XMIN,DX,YMIN AND DY USING 'SCALE' OF CIL
C
   10 CALL SCALE(X,NPTS,XLNTH,XMIN,DX)
      DO 7 I=2,NCOLS
      DO 8 K=1,NPTS
    8 Y(K)=P(K,I)
      CALL SCALE(Y,NPTS,YLNTH,Y1,Y2)
      IF(I.NE.2)GO TO 9
      YMIN=Y1
      DY=Y2
      GO TO 7
    9 IF(YMIN.GE.Y1)YMIN=Y1
      IF(DY.LE.Y2)DY=Y2
    7 CONTINUE
C
C        CALCULATING STARTING VALUES OF AXES IN CM
C
   11 XSTRT=0.0
      YSTRT=0.0
      IF(XMIN.GE.0.0)GO TO 4
      YSTRT=-XMIN/DX
    4 IF(YMIN.GE.0.0)GO TO 5
      XSTRT=-YMIN/DY
C
C        DRAWING AXES WITH BLUE PEN
C
    5 CALL CHPEN(BLUE)
      CALL AXIS(0.0,XSTRT,XLNTH,0.0,XMIN,DX,1H ,0)
      CALL AXIS(YSTRT,0.0,YLNTH,90.0,YMIN,DY,1H ,-1)
C
C        PLOTTING CURVES ON THE SKETCHED AXES WITH BLACK PEN
C
```

```
      CALL CHPEN(BLACK)
      DO 2 I=2,NCOLS
      J= 9-I
      DO 3 K=1,NPTS
    3 Y(K)=P(K,I)
      IF(I.NE.NCHN)GO TO 6
C
C         PLOTTING THE CHAINED CURVE FOR I=NCHN POINT BY POINT
C
      CALL PLOT(YSTRT,XSTRT,-3)
      XC=X(1)/DX
      YC=Y(1)/DY
      CALL PLOT(XC,YC,3)
      DO 12 L=2,NPTS
      XC=X(L)/DX
      YC=Y(L)/DY
   12 CALL DPLOT(XC,YC,0.1)
      CALL PLOT(-YSTRT,-XSTRT,-3)
      GO TO 2
    6 CALL LINE(X,Y,NPTS,4,J,XMIN,DX,YMIN,DY)
    2 CONTINUE
C
C         DEFINING A NEW ORIGIN (FOR PLOTTER) FOR NEXT CALL OF 'ANJUM'
C
      IPRK=ICALL/3
      IF(3*IPRK.EQ.ICALL)GO TO 20
C
C         MOVING NEW ORIGIN OUT OF THE PRESENT GRAPH REGION.
C
      CALL PLOT(0.0,YLNTH,-3)
      GO TO 21
C
C         MAKING A SOFT PARK AFTER EVERY 3 GRAPHS
C
   20 CALL PLOT(0.0,0.0,998)
   21 RETURN
      END
```

```
      DIMENSION A(100,5)
      DIMENSION IPAR(3)
      DATA IPAR/1,1,0/
C
C        INITIALIZING THE PLOTTER
C
      CALL PLOTS(IPAR)
      CALL FAM(1)
      NROWS=100
      XL=14.0
      YL=10.0
      DO 500 IGRAF=1,500
      READ(15,*)IMORE,NCOLS,NC
      WRITE(6,*)IMORE,NCOLS,NC
C
C        WHEN IMORE=0 GIVE DUMMY VALUES FOR THE OTHER VARIABLES ABOVE
C
      IF(IMORE.LE.0)GO TO 501
      DO 1 I=1,NROWS
      READ(15,100)(A(I,J),J=1,NCOLS)
1     WRITE(6,100)(A(I,J),J=1,NCOLS)
100   FORMAT(5G14.7)
      CALL ANJUM(IGRAF,A,NROWS,NCOLS,NC,XL,YL,1,0.0,0.0,0.0,0.0)
500   CONTINUE
C
C        MAKING A HARD PARK BEFORE STOPPING
C
501   CALL PLOT(0.0,0.0,1000)
      STOP
      END
```

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